Dynamic Risk-Sharing with Two-Sided Moral Hazard

Rui R. Zhao*
Department of Economics
University at Albany - SUNY
Albany, NY 12222, USA
Tel: 518-442-4760
Fax: 518-442-4736
E-mail: rzhao@albany.edu

November 27, 2006

Running title: Repeated Two-Sided Moral Hazard

*This paper grew out of my PhD dissertation submitted to the University of Minnesota in November 2001. I am indebted to Narayana Kocherlakota for his guidance. I thank Andy McLennan, Arijit Mukherji, Ket Richter, V.V. Chari, Harold Cole, Christ Phelan, Dirk Krueger, Ken Beauchemin, and Mike Jerison for helpful comments. I am especially grateful to three anonymous referees whose detailed and extremely helpful comments and suggestions helped me improve the paper significantly. I also thank seminar participants at University of Iowa, University of Minnesota, Queen's University, SUNY Albany, and 2002 North America Winter Meetings of the Econometric Society for comments. All remaining errors are my own.

Abstract

A group of risk-averse agents repeatedly produce a perishable consumption good; individual outputs are observable but efforts are not. The contracting problem admits a recursive formulation, and the optimal value function is the fixed point of a contraction mapping. When the agents can be punished to the full extent in a single period, every continuation contract of an optimal contract is itself optimal; the marginal utility ratio between one agent and another is a submartingale. The results imply that it is in general important to restrict an agent whose moral hazard constraint is binding from saving through another agent within the risk-sharing group. Limited commitment and long-run implications of optimal contracting are also examined.

Keywords: Risk sharing, Two-sided moral hazard, Hidden effort, Dynamic contracts, Consumption distribution.

JEL Classification: C7, D3, D8, J0

1 Introduction

This paper is a study of the dynamics of efficient consumption distribution in a production economy with hidden efforts. The economy I consider is populated with individuals who engage in risky production of a non-storable good using their efforts as the only inputs. The agents, having no access to outside insurance, must share the risks in their individual outputs. Optimal risk-sharing requires an agent who has high output today to transfer some of the output to an agent with low output in return for more consumption in the future; an agent having low output should do the opposite. The difficulty with this intertemporal transfer scheme is that it can adversely affect agents' incentives to work because efforts are private information. For this reason, optimal allocation must balance the needs of both risk-sharing and incentive provision. The purpose of this study is to investigate how this tradeoff affects the distribution of consumption across the individuals in the economy when *all* of them may face incentive constraints.

The model I study in this paper relaxes two important assumptions that underlie most of the existing work on dynamic insurance with private information (exceptions will be discussed later in this section). These are one-sided incentive constraints and exogenous risks. The former assumption is maintained in dynamic principal-agent models with hidden action (Rogerson [38], Spear and Srivastava [39], Atkeson [3], and Phelan and Townsend [32]), and both assumptions are stipulated in studies of dynamic insurance with private exogenous risks, such as endowment shocks (Townsend [44], Green [18], Thomas and Worrall [41], and Phelan [35]), preference shocks (Atkeson and Lucas [4]), or unemployment risks (e.g. Atkeson and Lucas [5]). These assumptions, although capturing some important aspects of private information, are at odds with many risk-sharing groups, such as worker-consumers in a closed economy or countries in a monetary union, where agents' efforts are productive and responsive to risk-sharing arrangements, and *all* agents may face moral hazard problems. This paper takes one step toward filling this gap.

Specifically, the model considered in this paper builds on Rogerson [37] and Spear and Srivastava [39] but extends their principal-agent model to allow for *two-sided* hidden effort; period-by-period aggregate resource constraints as in Atkeson and Lucas [4] and Wang [46] are also imposed to ensure a closed economy. This setup is particularly suitable for the study of consumption distribution when risk-sharing and the incentive to work are the primary concerns.

To characterize optimal allocation in this model, I formulate the problem as a recursive optimization problem using expected lifetime utility of one agent as the state variable. When agents can be punished to the full extent in a single period, optimal allocations are necessarily recursively optimal, i.e. the continuations of the long-term contract remain optimal from every period onwards. This property enables me to derive simple laws of motion (Euler equations) that govern the optimal evolution of consumption distribution and effort choices.

These laws of motion imply that the stochastic processes of agents' marginal utility ratios are submartingales, namely they increase over time in conditional expectation. To put this result into perspective, recall that the seminal work of Rogerson [37] showed that in the repeated principal-agent model the ratio of the principal's marginal utility to that of the agent is a martingale. My result therefore generalizes Rogerson's finding to multi-sided moral hazard with any finite number of agents: if an agent faces a moral hazard constraint then the ratio of this agent's marginal utility to that of another agent is a submartingale; if the agent is not constrained by moral hazard then the ratio is a martingale.

This characterization sheds light on an important aspect of optimal resource allocation with private information. As we know the fundamental difficulty caused by incentive constraints is that optimal allocations are generally incompatible with unregulated trade between agents; ex post, certain markets need be closed to keep agents at the ex ante optimum (Mirrlees [30]). The question is what types of trades and markets should be prohibited for an optimal allocation to be implementable in a market equilibrium or in some other system of resource allocations.

The answer to this question is provided by the submartingale characterization: at an output history where an agent i's incentive constraint is binding, if i is given the chance to borrow or save through another agent j on lending terms that make j just indifferent, then holding efforts unchanged agent i prefers to save rather than borrow. To understand better this result, it is useful to contrast it with the previous finding, by Diamond and Mirrlees [11] and Rogerson [37], in repeated one-sided moral hazard with a risk-neutral principal. These authors showed that if the agent is the only one facing an incentive constraint and the principal has the resources to insure the agent, then the agent should be prohibited from saving in *external* credit markets in order for optimal allocations to be implementable. This is not the case for the more general multi-sided moral hazard problem.

In particular, the result in this paper reveals an important distinction between external and *internal* credit markets in repeated multi-sided moral hazard. In the external market, not only do optimal allocations exhibit savings constraints but they also exhibit *borrowing* constraints. This is due to the aggregate resource constraints and risk aversion of all agents. In contrast, in the internal markets there generally exist restrictions on saving, i.e. the optimal contracts prevent an agent facing a moral hazard constraint from saving through other agents. This distinction between external and internal markets is not essential in the principal-agent model because both markets offer the agent the same lending terms.

The submartingale characterization also has strong normative implications for long-run distribution: inequality of consumption among agents will keep growing over time. This is in keeping with the findings in one-sided moral hazard (e.g. Green [18], Thomas and Worrall [41], and Atkeson and Lucas [4]). More precisely though, I show that in an optimal allocation along almost all sample paths the consumption distribution either converges to some perfect risk-sharing outcome (i.e. marginal utility ratio is constant across states) or drifts toward extreme inequalities with some agent consuming all the output. Moreover, if an agent's consumption diverges to its minimum level, his effort may somewhat paradoxically converge to the most productive incentive compatible level; conversely, the effort may converge to the minimum level if consumption approaches its maximum level. The result therefore displays an extreme form of tradeoff between efficiency and inequality, which seems to fit a casual observation of income distributions in some economies.

I also extend the above analysis to the situation where one or more of the agents have only limited commitment to long-term contracting, which is modeled by imposing a minimum level of acceptable lifetime utility, along the lines of Atkeson [3], Atkeson and Lucas [5], and Phelan [34]. The main characterization results continue to hold; in particular, optimal contracts are still recursive optimal, and as long as his commitment constraint is not binding an agent still faces savings constraints in the internal credit market. But when the commitment constraint is binding, the agent may be restricted from borrowing from the other agents because otherwise the optimal contract may have to promise him a utility below the minimum acceptable level.

The characterizations of optimal consumption allocation derived in this paper can potentially be tested empirically. For instance, the laws of motion derived in Section 4 generate testable hypotheses against the alternative assumption of one-sided moral hazard, thus providing a method, along the lines of Townsend [45] and Ligon [27], for detecting binding individual moral hazard constraints in a given risk sharing pool.

A methodological contribution of this paper is the development of a simple recursive formulation of the dynamic contracting problem with two-sided moral hazard. This is possible when at least one of the agents has an unbounded utility function therefore can be punished arbitrarily in every period. The idea is that such an agent can serve as a risk averse principal, possibly facing moral hazard, and the state of the system can be characterized using the other agent's lifetime utility alone. The optimal lifetime utility of the "principal" as a function of that of the agent satisfies a Bellman equation. Compared with one-sided moral hazard, an important feature of this functional equation is that the value function itself also appears in the constraints. Previous studies, such as Thomas and Worrall [42] and Wang [46], showed that the mapping that defines such a functional equation usually fails to be a contraction. By contrast, here the mapping *is* a contraction in the appropriate function space. This result provides a simpler algorithm for computing optimal allocations than working directly with the utility possibility set as in the standard recursive approach (Abreu et al. [1]).

Two other studies on dynamic contracting are closely related to this paper. Wang [46] studied dynamic risk-sharing between two agents who endure periodic hidden endowment shocks. It is the first study of dynamic insurance with two-sided hidden information. Although models of hidden information are sometimes "isomorphic" to models of hidden action, it is not the case for a small number of agents with an infinite time horizon. With hidden information a general mechanism would ask agents to report their types without necessarily revealing their reports to each other. This requires an analysis of infinitely repeated games with private monitoring, a task still facing some unresolved difficulties.[1] To get around of this problem, Wang considers a class of mechanisms in which agents' reports are always made public. He then formulates the contracting problem as a recursive optimization problem using one agent's lifetime utility as a state variable. Unfortunately, the characterization of the optimal value function as a fixed point of a Bellman equation faces some difficulties. The problem can be attributed to the assumptions in [46] that budgets must be balanced *and* utility functions are bounded; the latter assumption is implicit, as pointed out by Phelan [35].

The other related paper is Friedman [13]. As in this paper, it studies a model of dynamic risk-sharing with hidden action. However, the methods and focuses of the two studies are quite different. Friedman treats the problem as a planner maximizing a weighted social welfare function. In order to solve for the optimal contracts, he formulates the problem recursively using the utility weights of all agents as state variables. By contrast, I develop a simpler recursive formulation using the lifetime utility of *one* of the two agents as the state variable. Another important modeling difference is that Friedman's analysis relies on the first-order approach (e.g. Rogerson [38]) and is confined to a limited class of output distributions. By contrast, except for the independence and full support assumptions the analysis in this paper puts no further restriction on the output distributions; in particular it encompasses the popular case with finite actions. Moreover, while Friedman focuses on growing inequality in consumption over time, my analysis reveals the important implications of optimal risk-sharing on credit restrictions within and outside the group.

Finally, the stage model of this paper is obviously related to the existing studies of double moral hazard and moral hazard in teams. Aside from the fact that most

[1] See the Journal of Economic Theory (January 2002) symposium on repeated games with private monitoring for some recent progress.

of these studies deal with static contracting, another difference is that they mainly focus on the free rider problem due to joint production: when only total output is observed and agents share the output, it creates incentives for agents to shirk.[2] Note that this incentive problem will occur even if agents are risk neutral. In contrast, here I focus on the conflict between risk-sharing and the incentive to work, which arises even if agents' outputs are independent of each other.

The plan for the rest of the paper is as follows. Section 2 sets out the basic model and presents some preliminary analyses. Section 3 presents the recursive structure of optimal contracts. Section 4 characterizes optimal intertemporal allocations. Section 5 studies long-run behavior and presents a computed example. Section 6 concludes.

2 Statement of the Problem and Preliminary Analyses

In this section I spell out the details of the model economy, define optimal allocations, and state results related to the existence of optimal allocations.

Consider an economy with two long-lived agents, $i = 1, 2$. The analysis generalizes naturally to any finite number of agents. Time is discrete: $t = 1, 2, ...$ The physical environment remains unchanged at each date. Specifically, in every period agent i can produce a perishable consumption good using action (or effort) as the only input. The feasible actions of agent i are contained in some nonempty compact set $A_i \subseteq \Re$. Agent i's output θ_i is jointly determined by his action and some random shock. I assume that outputs θ_i take values in a finite set $\Theta_i \subseteq \Re_{++}$. Given action $a_i \in A_i$, output $\theta_i \in \Theta_i$ will be realized with probability $p_i(\theta_i|a_i)$.

For simplicity, I make the following assumption regarding distributions p_i.

A1. For $i = 1, 2$ probability function $p_i : A_i \times \Theta_i \to [0, 1]$ is continuous and $p_i(\theta_i|a_i) > 0$ for all $a_i \in A_i$ and all $\theta_i \in \Theta_i$.

The timing and information structure are as follows. In each period, first the agents simultaneously choose their actions, which are private information; next, outputs are realized and become public information; then according to pre-specified rules outputs may be transferred between the agents; finally consumption takes place at the end of the period (recall that the good is non-storable).

In this economy an allocation should specify agents' consumption and effort choices at each date based upon publicly available information. Specifically, the

[2] See Legros and Matsushima [25], Legros and Matthews [26], Bhattacharyya and Lafontaine [6], Al-Najjar [2], Gupta and Romano [20], and Kim and Wang [22] for double moral hazard, where both contracting parties can take hidden actions. See Holmstrom [21], Mookherjee [31], Lazear and Rosen [24], Demski and Sappington [10], and Ma [29] for moral hazard in teams, where a principal writes contract with a group of agents.

public information up to the end of date t is summarized in the *(public) history* $h^t = (\theta^1, ..., \theta^t)$, where $\theta^\tau \equiv (\theta_1^\tau, \theta_2^\tau)$ denotes date-τ output realizations of the two agents. It is convenient to let h^0 be the null history at the beginning of date 1.

A *consumption plan* c_i for agent i is a sequence of maps $\{c_i^t\}_{t=1}^\infty$ that specifies consumption $c_i^t(h^t)$ for every end-of-period-t history h^t.[3] Similarly an *action plan* s_i is a sequence of maps $\{s_i^t\}_{t=1}^\infty$ that specifies effort choice $s_i^t(h^{t-1})$ for every beginning-of-period-t history h^{t-1}.

Then an allocation or a *contract* $\sigma = (c_i, s_i)$ is a pair of consumption plans and a pair of action plans.

I assume that agent i has the following utility function

$$U_i(\sigma) = E \sum_{t=1}^{\infty} \delta^{t-1}[u_i(c_i^t) - g_i(s_i^t)]$$

where the expectation is taken with respect to the probability distribution, induced by the action plans (s_i), over all histories h^t.

Note that agents' utility functions in each period are additively separable in consumption and effort. This separability together with the following assumption will greatly simplify the analysis of this model.

A2. For $i = 1, 2$, the effort cost function $g_i : A_i \to \Re_+$ is continuous and $\min_{a_i} g(a_i) = 0$; the real-valued consumption utility function $u_i(\cdot)$ is defined on an interval $(\underline{c}_i, \infty)$ with $\underline{c}_i = 0$ or $-\infty$, $u_i'(\cdot) > 0$, $u_i''(\cdot) \leq 0$ (< 0 for some i), and $\lim_{c \to \underline{c}_i} u_i(c) = -\infty$.[4]

For the most part of the paper I assume that the lower bound $\underline{c}_i = 0$.[5] The assumption $u_i(\underline{c}_i) = -\infty$ means that the agents can be punished arbitrarily severely in a single period.

The following alternative assumption will also be considered.

A2'. The same as assumption A2 except that agent 1's consumption utility function u_1 is defined on the interval $[\underline{c}_1 = 0, \ \infty)$ hence $u_1(\underline{c}_1) > -\infty$.

Since actions are not observable, incentive compatibility must be imposed. Specifically, contract $\sigma = (c_i, s_i)$ is *incentive compatible* if for $i = 1, 2$, given the plans (c_1, c_2) and s_{-i},

$$U_i(c_1, c_2, s_i, s_{-i}) \geq U_i(c_1, c_2, \tilde{s}_i, s_{-i})$$

[3]For subsequent analysis it is often convenient to also consider the equivalent *utility plan* (z_i) that assigns utility $z_i(h^t) = u_i(c_i(h^t))$ to agent i at history h^t.

[4]The class of utility functions satisfying (A2) include the CRRA utility functions: $u(c) = c^{1-\gamma}/(1-\gamma)$ for $\gamma \geq 1$, in which case the consumption lower bound $\underline{c} = 0$, and the CARA utility functions: $u(c) = -e^{-\gamma c}$ for $\gamma > 0$, in which case $\underline{c} = -\infty$.

[5]However, see Section 6 Proposition 8 for a case where $\underline{c}_i = -\infty$.

for every action plan $\tilde{s}_i$ of agent i. In other words, given the consumption plans (c_i) agents' action plans (s_i) constitute a Nash equilibrium for the induced infinite-horizon dynamic game. It is well known that when output distributions have full support, which is assumed in A1, contract σ is incentive compatible if and only if it satisfies the following one-step incentive constraints for $i, j = 1, 2,\ i \neq j$ at every history h^t:

$$s_i(h^t) \in \arg\max_{a_i \in A_i} \sum_{\theta_i, \theta_j} p(\theta_i|a_i)p(\theta_j|s_j(h^t))[u_i(c_i(h^t, \theta)) + \delta U(\sigma|h^t, \theta)] - g_i(a_i).$$

Here $\sigma|h^t$ is the *continuation* contract of σ given history h^t.

Contract σ is *feasible* if it satisfies the resource constraints

$$c_1^t(h^t) + c_2^t(h^t) \leqslant \theta_1^t + \theta_2^t, \quad \forall\, h^t \neq h^0 \tag{1}$$

and the limited commitment constraints

$$U_i(\sigma|h^t) \geqslant \underline{U}_i, \quad \forall\, h^t,\ \forall\, i. \tag{2}$$

The resource constraints ensure that the economy is closed. The limited commitment constraints have the following interpretation: at the beginning of period t each agent i may choose to walk away from the ongoing contract and take some outside option thereafter. For instance, the agents may declare bankruptcy and go into autarky forever. I assume that agent i can obtain some expected lifetime utility no more than a fixed level $\underline{U}_i$ from the outside options, and if the agent does not walk away at the beginning of a given date then he would have to honor the current contract for that period.[6] I allow the possibility that $\underline{U}_i = -\infty$, in which case agent i does not face limited commitment constraints.

A feasible and incentive compatible contract is said to be *incentive feasible*. To make the problem nontrivial, I assume that each $\underline{U}_i$ is not too large so that there exist incentive feasible contracts; a sufficient condition for this is that $\underline{U}_i$ is no larger than the optimal autarkic payoff of agent i.

The aim of this paper is to study allocations that are Pareto optimal amongst all incentive feasible allocations.

Definition 1. A contract σ is *constrained Pareto optimal* (henceforth *optimal*) if σ is incentive feasible and there does not exist another incentive feasible contract σ' such that $U_i(\sigma') > U_i(\sigma)$ and $U_{-i}(\sigma') \geqslant U_{-i}(\sigma)$, for some $i = 1, 2$.

Every incentive feasible contract σ delivers a vector of lifetime utilities $(U_i(\sigma))$ to the agents. Such utility vectors constitute the *utility possibility set*. Its Pareto frontier comprises the utility vectors attainable by the optimal contracts.

[6] This is a formulation also adopted by Atkeson and Lucas [5] and Phelan [34].

In the remainder of this section, I will establish the existence of optimal contracts. For this and characterization purposes, it is useful to consider the utility frontier of agent 2.

Let $\mathcal{D}$ be the set of lifetime utilities of agent 1 that are attainable by incentive feasible contracts. For all $\xi \in \mathcal{D}$, define $\Sigma(\xi)$ as the set of incentive feasible contracts σ with $U_1(\sigma) = \xi$, and let $\mathcal{V}(\xi) = \{U_2(\sigma) \,|\, \sigma \in \Sigma(\xi)\}$ be the set of agent 2's lifetime utilities attainable by the contracts in $\Sigma(\xi)$. This defines a correspondence $\mathcal{V} : \mathcal{D} \twoheadrightarrow \Re$, whose graph is the utility possibility set.

Lemma 1. *If A1 and either A2 or A2' are satisfied, then* (a) *the correspondence $\mathcal{V} : \mathcal{D} \twoheadrightarrow \Re$ has a closed graph, i.e. for every sequence $(\xi^n, y^n) \to (\xi, y) \in \Re^2$ with $\xi^n \in \mathcal{D}$ and $y^n \in \mathcal{V}(\xi^n)$ for all n, we have $\xi \in \mathcal{D}$ and $y \in \mathcal{V}(\xi)$;* (b) *for every $\xi_i \in \Re$ there exists $K_j \in \Re$ such that if $U_i(\sigma) \geqslant \xi_i$ then $U_j(\sigma) \leqslant K_j$.*

Proof. See the Appendix. □

Define $V(\xi) = \sup \mathcal{V}(\xi)$, for all $\xi \in \mathcal{D}$. Function $V(\cdot)$ corresponds to the *utility frontier of agent 2*, which contains the Pareto frontier of the utility possibility set.

The following lemma shows that the sup in the definition of V is always attained, i.e. for each promised utility ξ to agent one there is a contract that maximizes agent two's utility.

Lemma 2. *If A1 and either A2 or A2' are satisfied, then* (a) *for all $\xi \in \mathcal{D}$, there is an incentive feasible contract σ with $U_1(\sigma) = \xi$ and $U_2(\sigma) = V(\xi)$;* (b) *function $V(\cdot)$ is upper semi-continuous.*

Proof. Part (a) follows because by Lemma 1, for all $\xi \in \mathcal{D}$ set $\mathcal{V}(\xi)$ is closed and bounded from above.

Let $(\xi^n) \to \xi \in \mathcal{D}$ with $\xi^n \in \mathcal{D}$ and $y^n = V(\xi^n)$ for all n. By Lemma 1, the sequence (y^n) is bounded from above and $\limsup(y^n) \in \mathcal{V}(\xi)$. It follows that $V(\xi) \geq \limsup(y^n)$; hence V is upper semi-continuous. □

The utility frontier $V(\cdot)$ may not be strictly decreasing and therefore all points in its graph may not be Pareto optimal. Nevertheless, since $V(\cdot)$ is upper semi-continuous, optimal contracts always exist, as shown in the following proposition.

Proposition 1. *Assume A1 and either A2 or A2'. Given $\xi_1, \xi_2 \in \Re$, if there is some $x \in \mathcal{D}$ and $y \in \mathcal{V}(x)$ with $x \geqslant \xi_1$, $y \geqslant \xi_2$, then there exists an optimal contract σ with $U_1(\sigma) \geqslant \xi_1$ and $U_2(\sigma) \geqslant \xi_2$.*

Proof. By Lemma 1, the intersection of graph of $\mathcal{V}$ and set $\{(x, y) \,|\, x \geq \xi_1,\, y \geq \xi_2\}$ is closed and bounded hence compact. Its projection, $\mathcal{D}' \equiv \{x \in \mathcal{D} \,|\, x \geq \xi_1, \text{and}\, \exists\, y \in \mathcal{V}(x)\, s.t.\ y \geq \xi_2\}$ is also compact. Then the upper semi-continuous function $V : \mathcal{D} \to \Re$ attains its maximum on set $\mathcal{D}'$. Moreover, the compact set of maximizers in $\mathcal{D}'$ has a maximal element x^*. A contract σ with $U_1(\sigma) = x^*$ and $U_2(\sigma) = V(x^*)$ clearly is optimal, because if there is any σ' with $U_1(\sigma') = \xi > x^*$ and $U_2(\sigma') = V(x^*) \geq \xi_2$ it contradicts the definition of x^*. $\square$

If both agents' consumption utility functions u_i are unbounded from below then $V(\cdot)$ is strictly decreasing and indeed defines the Pareto frontier.

Lemma 3. *If A1 and A2 are satisfied, then* (a) *function* $V(\cdot)$ *is strictly decreasing and continuous; and* (b) *contract* σ *is Pareto optimal if and only if* $U_2(\sigma) = V(\xi)$ *with* $\xi \equiv U_1(\sigma)$.

Proof. We only need prove part (a), since for (b), the "only if" part is obvious and the "if" part would follow if $V(\cdot)$ is strictly decreasing.

We first prove monotonicity. Let σ be an incentive feasible contract with $U_1(\sigma) = \xi \in \mathcal{D}$ and $U_2(\sigma) = V(\xi)$; let $\xi' \in \mathcal{D}$ with $\xi' < \xi$. If there is an incentive feasible contract σ' with $U_1(\sigma') = \xi'$ and $U_2(\sigma') > V(\xi)$, then $V(\xi') > V(\xi)$.

The contract σ' can be constructed from the initial contract σ by "reshuffling" agents' consumptions at date 1 without affecting agents' incentives to take the given actions.

Specifically, we reduce agent 1's expected utility conditional on each of his output realization θ_1, i.e. $\sum_{\theta_2} p(\theta_2|a_2)u_1(c_1(\theta_1, \theta_2))$, by the same amount $\xi - \xi' > 0$. To keep incentives of agent 2 intact, consumption should be transferred from agent 1 to agent 2 in a way so that for each θ_1 agent 2's utilities $u_2(c_2(\theta_1, \theta_2))$ are increased by some *equal* amount, $y(\theta_1) > 0$, across all θ_2.

In other words, we need only find $y(\theta_1)$ for all θ_1 such that agent 1's new consumptions

$$\widetilde{c}_1(\theta) = c_1(\theta) + c_2(\theta) - u_2^{-1}\left[u_2(c_2(\theta)) + y(\theta_1)\right]$$

satisfy

$$\sum_{\theta_2} p(\theta_2)u_1(c_1(\theta)) - \sum_{\theta_2} p(\theta_2)u_1(\widetilde{c}_1(\theta)) = \xi - \xi', \quad \forall \theta_1. \tag{3}$$

Note that the left-hand side of (3) is continuous and strictly increasing in $y(\theta_1)$, is equal to 0 if $y(\theta_1) = 0$, and goes to ∞ if $y(\theta_1)$ becomes large. Therefore there exists some $y(\theta_1)$ satisfying the equation for all θ_1.

By construction, each agent's expected utility is changed by an equal amount conditional on his own output realizations. Therefore the new contract is incentive compatible given that the initial contract is. Moreover, $U_1(\sigma') = \xi'$ and $V(\xi') \geq U_2(\sigma') > V(\xi)$. Thus $V(\cdot)$ is strictly decreasing.

Since $V(\cdot)$ is upper semi-continuous by Lemma 2, it only remains to prove that it is also lower semi-continuous. Consider some $\xi' > \xi$. We can construct contract σ' in the same way as in the above by finding $y(\theta_1) < 0$, $\forall \theta_1$, that satisfy equation (3). By construction, $V(\xi') \geq U_2(\sigma') = V(\xi) - \varepsilon$ with $\varepsilon = -\sum_{\theta_1} p(\theta_1|a_1)y(\theta_1) > 0$. By (3), $\xi' \downarrow \xi$ implies $y(\theta_1) \uparrow 0$ and hence $\varepsilon \downarrow 0$. Since $V(\xi') > V(\xi)$ for $\xi' < \xi$, this implies that $V(\xi) \leq \liminf_{\xi' \to \xi} V(\xi')$. Hence V is lower semi-continuous and the proof is complete. $\square$

3 A Recursive Formulation

In this section I show that the utility frontier $V(\cdot)$ satisfies a recursive functional equation. An algorithm then is developed to solve this functional equation. When both agents have unbounded utility functions, this recursive structure is strengthened to recursive optimality, which permits simple characterizations of the optimal contracts in the next section.

3.1 A Bellman Equation

I begin the analysis with a complete characterization of the domain of function V when neither agent faces a limited commitment constraint. Recall that this is the set of agent one's lifetime utilities attainable in incentive feasible contracts.

First, define a real number Q as follows. If the lower bound on agent 2's consumption $\underline{c}_2 = -\infty$, then let $Q = u_1(\infty)/(1-\delta)$. It is possible that $Q = \infty$.

If the lower bound $\underline{c}_2 = 0$ then define Q by the following program. If a_2 is *implementable*, i.e. if $\exists\, c(\theta_2) > 0$, $\forall \theta_2 \in \Theta_2$, such that

$$a_2 \in \arg\max_{\hat{a}_2 \in A_2} \sum_{\theta_2} u_2(c_2(\theta_2)) p_2(\theta_2|\hat{a}_2) - g_2(\hat{a}_2),$$

then define

$$Q(\lambda,\, a_2) \equiv \frac{1}{1-\delta} \max_{a_1 \in A_1} \sum_{(\theta_1,\theta_2)} u_1\left(\lambda(\theta_1+\theta_2)\right) p_1(\theta_1|a_1) p_2(\theta_2|a_2) - g_1(a_1) \tag{4}$$

where the parameter $\lambda \in (\underline{c}_1,\ 1]$ if assumption A2 holds and $\lambda \in [0,\ 1]$ if A2' holds. If a_2 is not implementable, let $Q(\lambda, a_2) = -\infty$.

Then let

$$Q = \max_{a_2 \in A_2} Q(1, a_2). \tag{5}$$

Program $Q(\lambda,\, a_2)$ has the following interpretation: at each date if agent 1 consumes a fraction λ of the total output and agent 2 is made to take action a_2 then agent 1's optimal payoff equals $(1-\delta)Q(\lambda,\, a_2)$. Since every implementable action

a_2 can be implemented with arbitrarily low resource costs, the payoff $Q(\lambda,\, a_2)$ can always be attained for all $\lambda \in (\underline{c}_1,\, 1)$. Therefore Q is the supremum of the incentive feasible lifetime utilities of agent one.

Note that an optimal solution to Problem (4) exists because set A_1 is compact and the objective function is continuous in a_1; the maximum in (5) is also well-defined because by the Theorem of the Maximum, function $Q(1, a_2)$ is continuous in a_2 and the set of implementable a_2's is compact.

Let $\mathcal{D}_\infty$ be the set of incentive feasible lifetime utilities of agent one when default payoffs $\underline{U}_i = -\infty$, $\forall i$. The following lemma shows that $\mathcal{D}_\infty$ is an interval with Q as the least upper bound.

Lemma 4. *Set $\mathcal{D}_\infty = (-\infty,\, Q)$ if Assumptions A1 and A2 are satisfied, and $\mathcal{D}_\infty = [0,\, Q)$ if A1 and A2' are satisfied.*

Proof. See the Appendix. □

Next I develop the recursive characterization of the utility frontier V. To this end, I define an operator T on the space of real-valued functions with domains $S \subseteq \mathcal{D}_\infty$ in the following functional equation.

(FE) For $f : S \subseteq \mathcal{D}_\infty \longrightarrow \Re$ and for all $\xi \in S$,

$$(Tf)(\xi) = \sup_{a_i,\, c_i(\cdot),\, U(\cdot)} \sum_{(\theta_1,\theta_2)} p(\theta_1|a_1)p(\theta_2|a_2)\big[u_2(c_2(\theta)) + \delta f(U(\theta))\big] - g_2(a_2) \tag{6}$$

subject to (7) to (10):

$$\sum_{\theta} p(\theta_1|a_1)p(\theta_2|a_2)\,[u_1(c_1(\theta)) + \delta U(\theta)] - g_1(a_1) = \xi \tag{7}$$

$$a_1 \in \arg\max_{a_1'} \sum_{\theta} p(\theta_1|a_1')p(\theta_2|a_2)\big[u_1(c_1(\theta)) + \delta U(\theta)\big] - g_1(a_1') \tag{8}$$

$$a_2 \in \arg\max_{a_2'} \sum_{\theta} p(\theta_1|a_1)p(\theta_2|a_2')\big[u_2(c_2(\theta)) + \delta f(U(\theta))\big] - g_2(a_2') \tag{9}$$

$$\begin{aligned} \forall\theta : c_1(\theta) + c_2(\theta) \leqslant \theta_1 + \theta_2,\ U(\theta) \in S \\ c_i(\theta) > \underline{c}_i \text{ (“}\geqslant\text{”for } i = 1 \text{ if A2' holds).} \end{aligned} \tag{10}$$

Note that Eq. (7) is the promise-keeping constraint for agent 1; (8) and (9) are the incentive constraints for agents 1 and 2 respectively; (10) is the feasibility constraints.

Functional equation (FE) and operator T have the following straightforward interpretation: if agent 1 must be promised continuation utilities $U(\theta)$ in set S

and agent 2 must stay on the continuation utility frontier f in the next period, then the current utility frontier of agent 2 is given by $T(f)$. For this reason, Tf is said to be *generated* by f.

The next two lemmas establish the link between the optimal value function V and functional equation (FE). Lemma 5 shows that it is sufficient to stay on the utility frontier $V(\cdot)$ to generate points on this frontier; Lemma 6 shows that V is a fixed point of T.

Lemma 5. *If A1 and either A2 or A2' are satisfied and σ is an optimal contract with $U_2(\sigma) = V(U_1(\sigma))$, then there exists another incentive feasible contract σ' with $U_i(\sigma') = U_i(\sigma)$ for $i = 1, 2$, and $U_2(\sigma'|\theta) = V(U_1(\sigma'|\theta))$ for all θ.*

Proof. See the Appendix. □

Lemma 6. *If A1 and either A2 or A2' are satisfied then the optimal value function $V : \mathcal{D} \to \Re$ is a fixed point of operator T: $T(V) = V$, and the sup in Program* (FE) *is attained at this fixed point.*

Proof. See the Appendix. □

These two lemmas imply that one can characterize the optimal value function V by solving the Bellman equation (FE). One feature of functional equation (FE) is that the continuation value function f appears in the constraints. This is also true in the models of Thomas and Worrall (1994) and Wang (1995). As shown by these authors, the mapping that defines such a functional equation may fail to be a contraction, which can make it difficult to compute the optimal value function.

What is new in this model is that the mapping T defined in (FE) *is* a contraction when the agents have limited commitments.

Specifically, let S be a compact subset of $\mathcal{D}_\infty$ and let $B(S)$ be the set of *bounded* real-valued functions defined on set S. Note that endowed with the sup metric $B(S)$ is a complete metric space.

The following proposition provides the basic characterization of the operator T on the space $B(S)$ and also offers a powerful algorithm for finding function V when its domain $\mathcal{D} = S$ is known.

Proposition 2. *Suppose that A1 and either A2 or A2' are satisfied. Then the operator T defined in (FE) is a contraction mapping on $B(S)$.*

Proof. See the Appendix. □

It is useful to know when the optimization problem in (FE) has a solution. Let $B^u(S)$ be the set of bounded real-valued upper semi-continuous functions defined on S. Note that $B^u(S)$ is a closed subset of $B(S)$.[7] Lemma 7 below, which parallels Lemma 2, shows that the sup in (FE) is attained for $f \in B^u(S)$ and $T(B^u(S)) \subseteq B^u(S)$; hence the fixed point of T is upper semi-continuous.

Lemma 7. *Suppose that A1 and either A2 or A2' are satisfied. If $f \in B^u(S)$ then* (a) *for all $\xi \in S$, there is a policy vector $(a_i, c_i(\cdot), U(\cdot))$ that satisfies the constraints of (FE) and attains $Tf(\xi)$;* (b) *function $Tf \in B^u(S)$.*

Proof. See the Appendix. □

In Proposition 2 and Lemma 7, agent one's expected utility is assumed to be within some compact set S. This is consistent with both agents having limited commitment constraints, i.e. $\underline{U}_i > -\infty \; \forall i$, because in this case the domain of function V is indeed a compact set $\mathcal{D} \subset \mathcal{D}_\infty$. If set $\mathcal{D}$ is known in advance then one can easily compute the optimal value function V using the contraction mapping algorithm.

However, when only the default payoffs $\underline{U}_i$ are given, the domain $\mathcal{D}$ is typically not known a priori and hence must be solved as part of the characterization of V.

The following proposition provides a method, based on the above contraction mapping algorithm, for computing V and its domain $\mathcal{D}$ at the same time.

Let S_0 be a compact subset of $\mathcal{D}_\infty$ that contains $\mathcal{D}$. Let f_0 be the unique fixed point of the operator T defined in (FE) on the space $B(S_0)$. For $k = 1, 2, ...,$ let

$$S_k = \{\, \xi \in S_{k-1} \mid \xi \geqslant \underline{U}_1 \text{ and } f_{k-1}(\xi) \geqslant \underline{U}_2 \,\}$$

and $f_k : S_k \to \Re$ be given by $f_k = T(f_{k-1}^k)$, where f_{k-1}^k is the restriction of f_{k-1} to set S_k.

Proposition 3. *Suppose that A1 and either A2 or A2' are satisfied. Then (S_k) is a nested decreasing sequence of compact sets that converges to $\mathcal{D}$, and for all $\xi \in \mathcal{D}$ the sequence $(f_k(\xi))$ is monotone decreasing and converges to $V(\xi)$.*

Proof. See the Appendix. □

Proposition 3 is analogous to the recursive algorithm proposed by Abreu et al. [1]. The difference is that instead of doing iteration on the utility possibility sets we work directly with the utility frontiers of the utility sets.

A computed example using the contraction mapping algorithm is presented in Section 5.

[7] Let (f_k) be a convergent sequence in $B^u(S)$ with limit $f \in B(S)$. Let $(\xi_n) \to \xi \in S$ with $\xi_n \in S$ for all $n \geq 1$ and $y = \limsup f(\xi_n)$. Fix $\varepsilon > 0$. Then for large n and k, $y < f(\xi_n) + \frac{\varepsilon}{4} < f_k(\xi_n) + \frac{\varepsilon}{2} < f_k(\xi) + \frac{3\varepsilon}{4} < f(\xi) + \varepsilon$.

In addition to providing a computational tool, functional equation (FE) also transforms the dynamic contracting problem into a static one, a study of which can reveal more characteristics of the optimal contracts. At this level of generality, however, the optimal value function V may not be differentiable everywhere and may contain non-concave portions, which make the problem in (FE) difficult to analyze using the standard methods. Of course one can allow public randomization in contracting so as to convexify the utility possibility set; then the value function V will be concave and (FE) will be a relatively simple concave programming problem. (See Phelan and Townsend [32] for an analysis of the repeated principal-agent problem with random contracts and Zhao [48] for a study of the renegotiation-proof contracts with randomization.)

But it turns out that the main results of this paper can be derived without introducing random contracts as long as Assumption A2 is satisfied. Therefore, to simplify the analysis I assume that A2 is satisfied in the rest of the paper. This will allow a much cleaner characterization of the optimal contracts.

3.2 Recursive Optimality

First, the recursive structure can be strengthened when both agents' utility functions are unbounded from below, as shown in the following proposition. This result provides the basis for further characterizations in the next section.

Proposition 4. *If A1 and A2 are satisfied, and contract σ is optimal then:*
(a) *every continuation contract $\sigma|h^t$ is optimal;*
(b) *total consumption always equals total output: $c_1(h^t) + c_2(h^t) = \theta_1^t + \theta_2^t$ for all $h^t \neq h^0$;*
(c) *there does not exist a history $h^t \neq h^0$ where one can find some consumption assignments $\hat{c}_i(h^t)$ and an incentive feasible contract $\hat{\sigma}$ such that $u_i(\hat{c}_i(h^t))+\delta U_i(\hat{\sigma}) \geq u_i(c_i(h^t)) + \delta U_i(\sigma|h^t)$ for $i = 1, 2$ with strict inequality for some i.*

Proof. We only need prove part (c), as it implies (a) and (b).

Suppose that contrary to the claim in (c), at some history $h^1 = \theta' = (\theta_1', \theta_2')$ there exist some consumption assignments $\hat{c}_i(\theta')$ and an incentive feasible contract $\hat{\sigma}$ such that $u_1(\hat{c}_1(\theta')) + \delta U_1(\hat{\sigma}) = u_1(c_1(\theta')) + \delta U_1(\sigma|\theta')$ and $d \equiv u_2(\hat{c}_2(\theta')) + \delta U_2(\hat{\sigma}) - (u_2(c_2(\theta')) + \delta U_2(\sigma|\theta')) > 0$; if agent 1 is strictly better off then one can reduce $\hat{c}_1(\theta')$ and increase $\hat{c}_2(\theta')$ to make these conditions satisfied. I shall show that σ fails to be optimal by finding a Pareto superior contract $\tilde{\sigma}$.

Contract $\tilde{\sigma}$ is constructed based on σ and $\hat{\sigma}$ as follows. First replace continuation contract $\sigma|\theta'$ with $\hat{\sigma}$. Since agent 2's expected utility conditional on θ' is increased by $d > 0$, his incentive constraints at date 1 may be affected.

To restore agent 2's incentives, I shall "reshuffle" agents' consumptions at date 1. For this purpose, it is convenient to consider the utility assignments

$z_i(\theta) = u_i(c_i(\theta))$ to the agents. I transfer consumptions from agent 2 to agent 1 at states (θ_1, θ_2'), $\forall \theta_1$, in such a way that agent 1's utility $z_1(\theta_1, \theta_2')$ is increased by an equal amount y for all θ_1 and agent 2's expected utility conditional on θ_2' is reduced by the amount $p_1(\theta_1')d$.

Namely, for θ_2' and for all θ_1, the new utility assignments of agent one at date 1 are given by

$$\tilde{z}_1(\theta_1, \theta_2') = z_1(\theta_1, \theta_2') + y$$

where $y > 0$ is chosen such that

$$\sum_{\theta_1} p_1(\theta_1)\tilde{z}_2(\theta_1, \theta_2') = \sum_{\theta_1} p_1(\theta_1) z_2(\theta_1, \theta_2') - p_1(\theta_1')d$$

which is

$$\sum_{\theta_1} p_1(\theta_1) u_2\big[c(\theta_1, \theta_2') - u_1^{-1}(z_1(\theta_1, \theta_2') + y)\big]$$
$$= \sum_{\theta_1} p_1(\theta_1) u_2[c(\theta_1, \theta_2') - u_1^{-1}(z_1(\theta_1, \theta_2'))] - p_1(\theta_1')d \qquad (11)$$

where $c(\theta_1, \theta_2') = c_1(\theta_1, \theta_2') + c_2(\theta_1, \theta_2')$ are the total consumptions in σ.

Note that the left-hand-side of (11) is a continuous, strictly decreasing function $L(y)$ with $L(0) >$ right-hand-side and $L(y) \longrightarrow -\infty$ as y increases. Therefore there exists a desired $y > 0$ that satisfies (11).

Compared with initial contract σ, each agent's expected utility conditional on his own output either remains unchanged or is increased by the same amount across his output realizations. Therefore the new contract $\tilde{\sigma}$ is incentive compatible. Since the reshuffles of consumptions always kept both agents' continuation utilities at least as large as in σ, the final contract $\tilde{\sigma}$ also satisfies the limited commitment constraints and hence is feasible. Moreover, we have $U_1(\tilde{\sigma}) > U_1(\sigma)$, $U_2(\tilde{\sigma}) = U_2(\sigma)$. Thus σ is not optimal, which is a contradiction.

If part (c) does not hold for some history h^t with $t > 1$, then applying the above arguments recursively one can prove that $\sigma|h^{t-1}, ..., \sigma|h^1$, and σ are not optimal. $\square$

Remark 1. Proposition 4 implies that when A1 and A2 are satisfied the results of this paper continue to hold if the resource constraints in (1) are equalities, i.e if the budget has to be balanced at all times. Such a situation arise naturally if the agents can not commit to burning outputs (c.f. Wang [46]).

Proposition 4 strengthens Lemma 5 in two ways. First, while Lemma 5 states that it is *sufficient* to stay on the utility frontier of agent 2 to generate points on the frontier, Proposition 4 goes further to show that optimal contracts are *necessarily* recursively optimal. As we will see in the next section this makes a big difference for the characterization of optimal contracts. Second, Proposition 4 shows that output burning is never needed in optimal contracts.

The intuition of Proposition 4, which is central to optimal risk-sharing in this model, can be summarized as follows. Suppose some continuation contract $\sigma|h^t$ is not optimal. To improve the situation, the obvious first step is to replace $\sigma|h^t$ with some Pareto superior contract. This of course may affect agents' incentives at date t or earlier. To restore the incentives at date t, we redistribute agents' consumptions in such a way that compared with σ one agent's expected utilities conditional on his own output signals remain unchanged and the other's utilities conditional on his own signals are increased by an equal amount across all signal values. The result is a superior incentive feasible continuation contract at h^{t-1}. Applying this procedure recursively towards period 1 and reshuffling consumptions alternately from one agent to the other will lead to a contract that Pareto dominates σ.

The above argument relies on several conditions: output signals are independent, preferences are separable, and consumption utility functions are unbounded from below. The separability conditions imply that each agent's incentive to work is determined by his expected payoffs conditional on his own output signals. Unbounded utility functions together with separable preferences take away the need for punishing the agents through ex post suboptimal continuation contracts.[8]

Recursive optimality also implies that optimal contracts are renegotiation-proof in a strong sense: if the agents are permitted to renegotiate the ongoing contract at the beginning of some date, they would be unable to find an incentive feasible contract that can result in Pareto improvement even if they can commit not to renegotiate any further in the future.[9] Since any future renegotiation will in general restrict the set of permissible contracts, recursive optimality therefore is the strongest renegotiation-proof concept that sticks to the Pareto principle, i.e. all agents have veto power.[10]

The recursive optimality of long-term optimal contracts when utility functions

[8]This assumption underlies the analyses in Grossman and Hart [19] and Rogerson [37]. Phelan and Townsend [32] assumes bounded utility function, which explains the use of suboptimal continuation contracts in optimal contracting there.

[9] Here renegotiation is assumed to occur only at the beginning of a period before any action is taken, which is consistent with the treatment in the repeated game literature. If renegotiation could occur after action is taken but before output signals are realized (c.f. [17]) then optimal contracts may not be renegotiation-proof. The underlying assumption here is that there is not enough time after actions are taken but before outputs are realized for renegotiation to take place.

[10]For instance here optimal contracts are strongly renegotiation-proof in the sense of Farrell and Maskin [12].

are unbounded is pointed out in Fudenberg, Holmstrom, and Milgrom [15] in the dynamic principal-agent model. Wang [47] studies renegotiation-proof contracts in the finitely repeated principal-agent model when the agent's utility function is bounded. Zhao [48] generalizes Wang's analysis to the infinitely repeated principal-agent model. Clementi and Hopenhayn [8] study renegotiation-proof debt contracts in a model of firm dynamics with financing constraints.

In the remainder of this section I will use functional equation (FE) to derive some properties of the contemporaneous pay-performance relation in optimal contracting. The question about optimal risk-sharing is to what extent will one agent's output affect the other's consumption. Lemma 8 below shows that the relationship between an agent's consumption and his own performance is to some extent immune to the other agent's performance.

Lemma 8. *Assume A1-A2. Fix an optimal contract and fix a period t. Let $u_i'(\theta_1, \theta_2)$ be the marginal utility of agent i when date-t outputs are (θ_1, θ_2). Then for $i, j = 1, 2$ with $i \neq j$, for all θ_i, θ_i', either $\frac{u_i'(\theta_i,\theta_j)}{u_j'(\theta_i,\theta_j)} \geq \frac{u_i'(\theta_i',\theta_j)}{u_j'(\theta_i',\theta_j)}$ for all θ_j or $\frac{u_i'(\theta_i,\theta_j)}{u_j'(\theta_i,\theta_j)} \leq \frac{u_i'(\theta_i',\theta_j)}{u_j'(\theta_i',\theta_j)}$ for all θ_j.*

Proof. See the Appendix. □

This result says that as one runs through agent i's output signals the sign of the change in the marginal utility ratio is independent of (or perfectly positively correlated across) the output realizations of agent j. In other words, to the extent that the marginal utility ratio reflects agent i's relative share of the total output, its variation, at least qualitatively, is primarily determined by the agent's own performance. It is in this sense that an agent's pay-performance relation is largely isolated from the other agent's performance. This of course is a reflection of imperfect risk-sharing, which is needed for motivating effort.

Next consider the optimization problem in (FE) when value function V replaces f. Note that by Proposition 4 agent 2's consumption is given by $c_2(\theta) = \theta_1 + \theta_2 - c_1(\theta), \forall\, \theta$. For a given promised utility ξ to agent 1, let $(a_1^*, a_2^*, c_1^*(\cdot), U^*(\cdot))$ be an optimal solution to program (FE). It then follows that given the vector $(a_1^*, a_2^*, U^*(\cdot))$, consumption choices $c_1^*(\cdot)$ maximize the objective function subject to the constraints. If action sets A_1, A_2 are both finite, these constraints will involve only finitely many inequalities. Then the problem can be analyzed using the standard Kuhn-Tucker conditions. With the help of the following additional assumption, a positive pay-performance relation will emerge.

A3. (Monotone likelihood ratio property) For $i = 1, 2$ and for $a_i, a_i' \in A_i$, $g_i(a_i) > g_i(a_i')$ implies that $p_i(\theta_i|a_i')/p_i(\theta_i|a_i)$ weakly decreases in θ_i.

Lemma 9. *Assume A1-A3 and that sets A_1 and A_2 are finite. If the binding incentive constraints for agent i involve only efforts a_i with $g_i(a_i) < g_i(a_i^*)$ then agent i's consumption $c_i^*(\theta_i, \theta_j)$ is nondecreasing in his output θ_i for all θ_j.*

Proof. See the Appendix. □

4 The Dynamics of Optimal Allocations

This section presents the main characterizations of the optimal allocations. I will focus on the dynamics of optimal consumption allocations and will show in the next section that the results also generalize to expected utilities and effort choices.

4.1 The Main Characterizations

The characterization of optimal consumption allocations hinges on the behavior of the marginal utility ratio $u_1'(c_1^t)/u_2'(c_2^t)$. In this model the first-best optimal risk-sharing would require the marginal utility ratio be constant both across contemporaneous output realizations and over time.

In constrained optimal contracts some variation in the marginal utility ratio is needed for motivating efforts. Nevertheless, optimal contracts attempt to smooth the variations in the marginal utility ratio both across states and over time by utilizing the intertemporal substitutability between current and future consumptions. This kind of consideration gives rise to close-form Euler-type conditions, given in the following proposition, that link the current consumption allocation (c_i^t, c_j^t) with the next-period output-contingent consumptions $(c_i^{t+1}(\theta_i, \theta_j), c_j^{t+1}(\theta_i, \theta_j))$.

Proposition 5. *Assume A1-A2. If the limited commitment constraints are nonbinding for both agents at history h^t, $t \geq 1$, then for $i \neq j$,*

$$\frac{u_i'(c_i^t)}{u_j'(c_j^t)} = E_{\theta_j}\left[E_{\theta_i}\frac{u_j'(c_j^{t+1}(\theta_i, \theta_j))}{u_i'(c_i^{t+1}(\theta_i, \theta_j))}\right]^{-1} \tag{12}$$

where E_{θ_q} is the expectation operator with respect to output signal θ_q, for $q = i, j$, given the efforts at date $t+1$.

Note that if the default payoffs $\underline{U}_i = -\infty$, $\forall i$, then the limited commitment constraints never bind and the result always holds.

The logic of Proposition 5 is at the heart of optimal risk-sharing and can be summarized as follows. Given any optimal contract, one can always think of "nearby" consumption plans that perturb the original consumption allocations but preserve the effort incentives. Such perturbations must be weakly Pareto inferior by recursive optimality. The equations given in Proposition 5 are necessary conditions of this fact.

Proof. I will prove the equation for $i = 2$, $j = 1$; the other case is symmetric.

Let σ be an optimal contract and let h^t, $t \geq 1$, be a period-t history. For each agent i, denote the consumption assignments at h^t by c_i^t and the corresponding utility assignments by z_i^t; for each output realization (θ_1, θ_2) in period $t+1$, denote the consumption assignments by $c_i(\theta_1, \theta_2)$ and the corresponding utility assignments by $z_i(\theta_1, \theta_2)$.

I shall construct a class of perturbed continuation contracts $\tilde{\sigma}|h^t$ by modifying the original consumption distributions at h^t and in period $t+1$. These perturbations are indexed by the real numbers $(\eta, \varepsilon, \nu(\theta_1))$ as follows.

First, at history h^t I increase agent 2's utility by a small amount η and decrease agent 1's consumption accordingly so as to satisfy the resource constraint. Namely,

$$\tilde{z}_2^t = z_2^t + \eta$$
$$\tilde{z}_1^t = u_1\Big(x_t - u_2^{-1}(\tilde{z}_2^t)\Big),$$

where x_t is the total output at history h^t. (By Proposition 4 agents always consume the total output.)

Next, I decrease agent 2's utility at date $t+1$ by η/δ so as to keep his expected utility at h^t unchanged. This, however, should be done without affecting either agent's incentives at h^t. To this end, for each of agent 1's output realization θ_1 and for all pairs (θ_1, θ_2), decrease agent 2's utility by some amount $\nu(\theta_1)$, i.e.

$$\tilde{z}_2(\theta_1, \theta_2) = z_2(\theta_1, \theta_2) - \nu(\theta_1)$$

and transfer the corresponding consumption to agent 1, i.e.

$$\tilde{c}_1(\theta) = x(\theta) - u_2^{-1}(\tilde{z}_2(\theta))$$

where $x(\theta) = \theta_1 + \theta_2$ again is the total output.

Note that the change in agent 2's utility only depends on agent 1's output. Moreover, the $\nu(\theta_1)$'s are chosen to satisfy the following two conditions:

- agent 2's expected utility in period $t+1$ is decreased by η/δ, i.e.

$$\sum_{\theta_1} p_1(\theta_1)\nu(\theta_1) = \frac{\eta}{\delta}, \tag{13}$$

- agent 1's expected utility in period $t+1$ conditional on each θ_1 is increased by an equal amount ε, i.e.

$$\begin{aligned} & \sum_{\theta_2} p_2(\theta_2)\; u_1\big[x(\theta) - u_2^{-1}(z_2(\theta_1, \theta_2) - \nu(\theta_1))\big] \\ = \;& \sum_{\theta_2} p_2(\theta_2)\; u_1\big[x(\theta) - u_2^{-1}(z_2(\theta_1, \theta_2))\big] + \varepsilon, \quad \forall\, \theta_1 \in \Theta_1. \end{aligned} \tag{14}$$

Note that the new consumption allocations are constructed respecting the resource constraints; and the limited commitment constraints should remain unaffected for small perturbations. Also, both agents' incentives at h^t and at dates beyond t should remain intact since agent 1's expected utility in period $t+1$ conditional on each of his own output signal θ_1 is increased by the same amount ε, and agent 2's expected utility in period $t+1$ is changed in a way that is independent of his output θ_2.

Moreover, compared with the original contract agent 2's expected utility at history h^t is unchanged in the perturbed contracts. Therefore, by part (c) of Proposition 4 agent 1 must be weakly worse off in the perturbations.

In other words, $(\varepsilon = 0, \eta = 0, (\nu(\theta_1) = 0))$ is an optimal solution to the following problem:

$$\max_{\varepsilon, \eta, (\nu(\theta_1))} u_1[x_t - u_2^{-1}(z_2^t + \eta)] + \delta\varepsilon \tag{15}$$
$$\text{s.t. (13) and (14).}$$

In the Appendix Eq. (12) is shown to be a necessary condition of this fact. □

Remark 2. If agent j's moral hazard constraint is slack at history h^t, then the problem reduces to the principal-agent problem with agent j being the risk averse principal. Then in the above variation argument we only need to perturb agent i's utility at h^t by η and his utility in period $t+1$ by η/δ for all output realizations; Eq. (12) then becomes:

$$\frac{u_j'(c_j^t)}{u_i'(c_i^t)} = E_t \frac{u_j'(c_j^{t+1})}{u_i'(c_i^{t+1})}.$$

This is the famous martingale result of Rogerson [37]. An intuitive way of seeing this directly from (12) is to suppose that agent j has only one output signal and hence does not have a moral hazard problem, then equation (12) clearly reduces to the above martingale equation. Therefore Proposition 5 generalizes Rogerson's result to the two-sided moral hazard model.

Note that with two-sided moral hazard constructing incentive-preserving perturbations of the optimal contract is more delicate: the perturbations must respect the incentive constraints of *both* agents. But that is not all: there is an even more subtle point regarding whether or not the variation method is applicable when the utility functions are bounded.

Obviously, variation works only if the consumption levels are in the interior. As long as the utility functions are unbounded (even if consumptions are bounded), this condition is automatically satisfied. However, if the utility functions are bounded then the argument may not work even if consumptions are in the interior.

In this case the validity of the variation method will depend upon whether moral hazard is two-sided or one-sided, as I explain next.

The central idea of the variation method rests on the condition that a small perturbation of the optimal contract is weakly inferior for one agent subject to the other agent being indifferent. In the principal-agent model, this is always true even if the agent's utility function is bounded from below: if a contract is optimal for the principal given the promised utility to the agent then any continuation contract is *necessarily* optimal for the principal given the promised continuation utility to the agent; otherwise we can replace the suboptimal continuation contract with a better one and obtain a better overall contract for the principal. Therefore, the variation argument can go through as long as the current and the next-period consumptions are all in the interior.

This logic, however, does not work when there is two-sided moral hazard, because replacing a continuation contract with a "better" one can affect the principal's incentives, which makes the perturbation invalid. Put it differently, according to Lemma 5 if agent 1's utility is bounded from below then it is only *sufficient* to stay on the utility frontier of agent 2 (the "principal") to generate the points on that frontier. Therefore it is not necessarily true that a perturbed continuation contract must be weakly inferior to the original one, i.e. Proposition 4 may not hold.[11] Then the variation method is not guaranteed to work even if the current and the next-period consumptions are all in the interior. Nevertheless, the method does apply and the results are the same if the agents' consumptions have been in the interior *since period 1*; in this case the reshuffling procedure can be applied recursively from period t back towards period 1, just like in the case of unbounded utility functions.

The next result, which follows from Proposition 5 directly, states that the sequence of marginal utility ratios is a *submartingale*. This result will be essential for studying the long-run behavior of optimal contracts.

Corollary 1. *Assume A1-A2. Given an optimal contract, if at history $h^t \neq h^0$ agent i's limited commitment constraint is nonbinding then marginal utility ratios satisfy*

$$\frac{u_i'(c_i^t)}{u_j'(c_j^t)} \leq E_t \frac{u_i'(c_i^{t+1}(\theta_i, \theta_j))}{u_j'(c_j^{t+1}(\theta_i, \theta_j))} \tag{16}$$

where the expectation is taken conditional on h^t, and the inequality is strict if there is some θ_j such that the ratio $u_i'(c_i^{t+1})/u_j'(c_j^{t+1})$ is not constant across all θ_i.

Proof. See the Appendix. □

[11] Of course if such a better continuation contract exists it has to destroy incentive constraints in period t or earlier.

The submartingale result rules out autarky as an optimal allocation.

Corollary 2. *Assume A1-A2. A sequence of stationary one-period contracts can not be optimal unless it is first best, i.e. unless marginal utility ratio is constant. In particular, autarky is not optimal.*

Proof. The argument is straightforward. Suppose to the contrary an optimal contract consists of a sequence of stationary one-period contracts. But if the one-period contract is not first best, i.e. u'_j/u'_i is not constant across all output realizations, then it contradicts the submartingale condition: $\max\{u'_j/u'_i\}$ can not be less than or equal to the expected value of u'_j/u'_i. Autarky is not optimal since it is stationary and is not first-best. □

Remark 3. The above results can be generalized easily to more than two agents. Specifically, the equation in Proposition 5 becomes: for all $i \neq j$,

$$\frac{u'_i(c^t_i)}{u'_j(c^t_j)} = E_{\theta_j}\left[E_{\theta_{-j}}\frac{u'_j(c^{t+1}_j(\theta_j,\theta_{-j}))}{u'_i(c^{t+1}_i(\theta_j,\theta_{-j}))}\right]^{-1} \tag{17}$$

where θ_{-j} denotes a generic vector of output signals produced by all agents other than agent j. The submartingale equation also holds:

$$\frac{u'_i(c^t_i)}{u'_j(c^t_j)} \leq E_t\frac{u'_i(c^{t+1}_i(\theta^t))}{u'_j(c^{t+1}_j(\theta^t))}. \tag{18}$$

And if agent i's moral hazard constraint is slack then

$$\frac{u'_i(c^t_i)}{u'_j(c^t_j)} = E_t\frac{u'_i(c^{t+1}_i(\theta^t))}{u'_j(c^{t+1}_j(\theta^t))}. \tag{19}$$

The proofs for these results are virtually the same as in the two-agent case. First, the recursive optimality as in Proposition 4 can be proved for the general case: we can show in much the same way that holding constant other agents' expected utilities it is not possible to improve agent i's utility without hurting agent j. Then in the variation argument we treat the pair of agents i and j under consideration separately from the rest of the agents as if it is a two-agent relationship; in particular, all other agents' consumptions remain unchanged.

Note that conditions (12), (16), and (17)-(19) hold for every optimal contract independent of the initial lifetime utilities of the agents. This is important for empirically testing these restrictions since data on initial utilities in general are not available. Specifically, the method developed by Ligon [27], who tested Rogerson's repeated principal-agent model using data from three villages in South India, can be adapted to test the hypothesis of two-sided moral hazard against the assumption of one-sided moral hazard, using the submartingale equation (18) vs. the martingale equation (19).[12]

[12] Details for implementing the test are available upon request.

4.2 Credit Markets and Savings Constraints

It is well-known that in order to implement optimal allocations it is essential to prevent the agents from participating in other credit markets. The interesting question is what types of restrictions are really necessary.

Diamond and Mirrlees [11] and Rogerson [37] have shown that in the repeated principal-agent model preventing the agent from *saving* is necessary for implementing the optimal allocations. To see this, note that with one-sided moral hazard and a risk neutral principal, the agent's marginal utilities satisfy $u'(c_t) \leq E_t u'(c_{t+1})$.[13] Since the agent places a greater value on the future consumptions than on the current consumption, he would wish to save his income at the fixed interest rate $r = 1/\delta - 1$. Therefore preventing the agent from saving is essential for implementing the optimal allocations.

To draw comparison with this seminal result, in this model it is important to distinguish between "external" and "internal" credit markets. External credit markets allow agents to borrow or save outside their resource constraints; internal credit markets on the other hand only allow the agents to borrow or save within the group. In the principal-agent model with a risk-neutral principal, the distinction between the two markets disappears because the agent would get the same lending terms whether he deals with the principal or an outside lender; as a result, the agent would face savings constraints in both markets. This is no longer true in the current environment.

I shall show that optimal allocations may exhibit both savings and borrowing constraints in the external market, but they always feature savings constraints in the internal market.

Consider first the external market. In this market the aggregate resource constraints and the risk aversion of both agents play a bigger role than the incentive constraints. For instance, suppose that the incentive constraints are all slack and risk-sharing is at the first-best (i.e. marginal utility ratio u_i'/u_j' is constant across states and over time). Suppose at some history the agents are suddenly allowed to borrow or save at the fixed interest rate $r = 1/\delta - 1$. Clearly if the total output is at the lowest level then agents' consumptions are at their lowest levels; therefore both agents would wish to *borrow* against their future incomes because their current marginal utilities are at their highest levels: $u_i'(c_t) > E_t u_i'(c_{t+1})$. On the other hand, if the total output is at the highest level then both agents would like to save because $u_i'(c_t) < E_t u_i'(c_{t+1})$. Therefore, it appears that to implement optimal allocations both borrowing *and* saving should be prevented in the external credit markets. However, this observation should be interpreted cautiously. If there were

[13]This can be seen from Eq. (19): with u_i' being constant, the result follows from Jensen's inequality.

a risk-neutral outsider who does not have a moral hazard problem, does not face resource constraints, and can offer insurance to the agents, then there would be no need for the agents to share the risks among themselves. Then the optimal allocation in the current model would not be optimal in the first place.

Consider next the internal credit market. Imagine that at some date after the outputs are realized the agents are suddenly allowed to borrow or lend between each other. In particular, suppose that agent 1 has all the bargaining power in the sense that agent 1 may borrow (or save) some income worth one unit of agent 2's utility and repay (or collect) an amount tomorrow worth $1/\delta$ units of agent 2's next-period utility so as to leave agent 2 indifferent. In other words, agent 2's utility is used as the "currency" in this internal credit market, and agent 1 can borrow or save at the fixed interest rate $r = \frac{1}{\delta} - 1$ using this currency.

Proposition 6. *At any history h^t where the submartingale characterization (18) holds for $i = 1$ and $j = 2$, holding efforts unchanged if agent 1 is given the above opportunity to borrow or save in the internal credit market through agent 2 then he would prefer to save, and he would strictly prefer to save if (18) holds with strict inequality.*

Proof. If agent one saves some income worth one unit of agent two's utility then the change in agent one's present expected utility is given by $-u_1'(c_1^t)/u_2'(c_2^t)$;[14] when agent one receives an amount worth $1/\delta$ units of agent two's utility in the next period the additional change in present expected utility is $E_t[u_1'(c_1^{t+1})/u_2'(c_2^{t+1})]$. In total the change in agent one's present expected utility is equal to

$$E_t \frac{u_1'(c_1^{t+1})}{u_2'(c_2^{t+1})} - \frac{u_1'(c_1^t)}{u_2'(c_2^t)}$$

which is non-negative by the submartingale condition (18). Therefore the marginal return of saving is nonnegative, hence agent one always has a (weak) desire to save some of his income through agent 2 given the effort choices. Obviously agent one strictly prefers to save if (18) holds with strict inequality.[15] $\square$

Note that the apparent Pareto improvement resulted from the savings arrangements in the internal market is not inconsistent with recursive optimality, because this "improvement" is conditional on agents choosing the efforts specified in the original contract. Since the agents can freely adjust their effort choices, the restriction to saving may not be really necessary for implementing optimal allocations,

[14]Agent one's utility as a function of u_2 is given by $u_1 = u_1(c_1 + c_2 - u_2^{-1}(u_2))$ hence its derivative is equal to $-u_1'(c_1)/u_2'(c_2)$.

[15]By Eq. (17), this inequality is strict if $u_1'(c_1^{t+1})/u_2'(c_2^{t+1})$ can take on two different values across output signals θ_{-2} for some signal θ_2.

at least if there are only two agents: if they know they have such savings opportunities the agents will find the efforts unattractive, and when they fully adjust their efforts accordingly their ex ante welfare can not be improved in a Pareto sense because of recursive optimality.

However, if the group has more than two agents it can not be ruled out that two of them can collude and engage in such savings arrangements at the expense of the others, and therefore it may be necessary to restrict the agents from such saving activities for the sake of implementing the optimal allocations. This consideration is also relevant if the agents are able to commit to the effort plans at the beginning. Such commitments do not affect the ex ante optimal contracts since incentive compatibility only requires the effort plans constitute a Nash equilibrium anyway; but in this case in order to implement optimal allocations savings in the internal credit market need be restricted even if there are only two agents.

By equation (19) and the preceding argument, an agent whose moral hazard and limited commitment constraints are both nonbinding does not face a savings or a borrowing constraint in the internal credit market.

In summary, putting aside the limited commitment issue, the robust finding is that whenever an agent has a binding moral hazard constraint he faces savings constraint in the *internal* credit market within the risk-sharing group. Viewed through this lens, the result of [11] and [37] for the principal-agent model is a special case of our result for the two-sided moral hazard model.

Next, consider what happens when the limited commitment constraint *is* binding. If there does not exist first-best contracts for the static contracting problem then we have the following result.

Lemma 10. *Assume A1-A2, $\underline{c}_i = 0$, $\forall i$, and that no first-best contract exists for the static problem. If $\underline{U}_1 > -\infty$ then for every optimal contract σ there is some history h^t at which agent 1's limited commitment constraint is binding, i.e. $U_1(\sigma|h^t) = \underline{U}_1$, and holding efforts unchanged agent 1 strictly prefers to borrow from agent 2 under the same lending terms as in Proposition 6.*

Proof. See the Appendix. ☐

Thus when his limited commitment constraint is binding an agent may face borrowing rather than savings constraint. This restriction however is of a different nature and should not be interpreted on the par with the savings constraint. From the efficiency point of view the restriction on borrowing is unnecessary, and in fact counterproductive. It exists only to prevent the promised utility from falling below $\underline{U}_i$. Put it differently, it is not the efficiency consideration that imposes the restriction on borrowing, rather it is the exogenous limited commitment constraint that makes such restriction necessary: allowing borrowing will give agent i the chance to take the money and run, because at such a history his promised payoff would fall below the default payoff $\underline{U}_i$ after he borrows from j. On a different note though, this distinction between borrowing and savings constraints provides a means to test whether an agent's limited commitment constraint is binding.

Finally, note that in the above discussion the borrowing and saving activities are assumed to be observable and thus can be controlled by the mechanism. Allowing private storage along the lines of Cole and Kocherlakota [9] would be an interesting avenue for future research.

5 The Long-Run Behavior of Optimal Allocations

In this section I examine the behavior of optimal contracts in the long run. The analysis generalizes the existing findings for one-sided moral hazard and for hidden information economies.

First, the submartingale characterization has strong implications for the long-run distribution, as given in the following proposition. Recall that when the agents do not face limited commitment constraints the set of agent i's incentive feasible lifetime utilities is given by $(-\infty,\ Q_i)$, where Q_i is determined by Program Q (letting agent i assume the role of agent 1) in Section 3.

Proposition 7. *Assume A1-A2 and that default payoffs $\underline{U}_i = -\infty$, $\forall\, i$. We have,*

(a) *for every optimal contract, along almost all history path exactly one of the following two situations occurs:* (i) *the consumption/utility distributions converge to some first-best outcome;* (ii) *each agent i's expected lifetime utility diverges to $-\infty$ or converges to Q_i or fluctuates between them;*

(b) *if the consumption lower bound $\underline{c}_2 = 0$ and agent 1's expected utility converges to Q_1, then agent 2's expected utility diverges to $-\infty$ and agents' effort choices converge to some $(\hat{a}_1, \hat{a}_2)$ that solve Program Q_1. Moreover, if the probability distributions $p_2(\cdot|a_2)$ satisfy the first-order stochastic dominance condition then $\hat{a}_2$ is the most productive implementable effort of agent 2;*

(c) *if $\underline{c}_j = -\infty$ and agent i's expected utility converges to $Q_i < \infty$ then agent i's effort converges to the minimum level $\underline{a}_i$.*

Proof. See Appendix. □

The results in part (a) are somewhat sharper than that in Wang [46], where it is only shown that agents' expected utilities do not converge to any point in the feasibility set but leaves open where they might go in the long run. Here it is shown that when the incentive problem never goes away even if agents' expected utilities may not diverge for sure to the ends of the feasible sets, they almost surely will fluctuate towards these ends.

According to part (b), in the long run an agent may get immiserated but yet is induced to choose his most productive incentive compatible effort. Therefore optimal contracts generate a subgroup of "working poor" in the long run. Part (c) describes the opposite possibility – an agent with sufficiently high utility entitlement may become a member of the "leisure class."

In light of part (a) of Proposition 7, to focus on the effects of moral hazard I make the following assumption, which ensures that the moral hazard problem is always present.

A4. In the static problem there does not exist a first-best contract in which the marginal utility ratio is constant across output realizations.

The next proposition deals with the situation when agents' marginal utilities satisfy certain boundary conditions, in which case the submartingales become bounded and convergence can be obtained.

Proposition 8. *Assume A1-A2, A4, and* $\underline{U}_i = -\infty$ *for both* i*. For* $i, j = 1, 2$ *with* $i \neq j$*, let* M_i *be the supremum of marginal utility ratio* $u_i'(c_i)/u_j'(c_j)$ *for feasible consumptions* (c_i, c_j)*. We have,*

(a) *if* $M_j = \infty$ *and only agent* i *has an moral hazard problem then agent* i*'s consumption and utility diverge to* $-\infty$ *in the long run;*

(b) *if* $M_i < \infty, M_j = \infty$ *and only agent* j *has incentive problem then both agents' consumptions and utilities diverge to* ∞ *or* $-\infty$ *with positive probabilities respectively;*

(c) *if* $M_i < \infty$ *for* $i = 1, 2$ *then both agents' consumptions and expected utilities diverge almost surely to* ∞ *or* $-\infty$ *with positive probabilities respectively.*

Proof. See Appendix. □

Note that given assumption A2 (unbounded utility functions), if consumption lower bound $\underline{c}_i = 0$ then $u_i'(\underline{c}_i) = \infty$ hence $M_i = \infty$. Therefore, part (b) and part (c) are applicable only if $\underline{c}_i = -\infty$ for at least one i. For instance, (b) applies to the situation when $u_1'(-\infty) < \infty$, $u_2'(\infty) > 0$, and either $u_1'(\infty) = 0$ or $u_2'(-\infty) = \infty$; and (c) applies to the situation when $u_i'(-\infty) < \infty$, $u_i'(\infty) > 0$ for both i. These cases arise naturally when the agents have exponential utility functions or one of them is risk neutral, and have been widely studied in the dynamic contracting literature.[16] It may be interpreted as agents taking on large amounts of debt from each other.

Proposition 8 generalizes some of the existing findings in the one-sided moral hazard models, including the repeated insurance problem between a risk neutral principal and a risk averse agent who is subject to hidden endowment shocks

[16]For instance, see Green [18], Atkeson and Lucas [4], and Phelan [35].

(Thomas and Worrall [41]; Green [18]) and the insurance problem between a planner and a continuum of risk averse agents (Atkeson and Lucas [4]). In these models, efficient allocations have the feature that the agents are immiserated in the long run.

In the current model of hidden action this can be seen heuristically from the submartingale characterization. With one-sided moral hazard and a risk neutral principal, the agent's marginal utilities satisfy

$$u'(c_t) \leq E_t u'(c_{t+1}). \tag{20}$$

Thus in optimal allocation the agent's consumption and utility are *front-loaded*, which explains why the agent gets immiserated in the long run. As Phelan [35] points out, however, this result also depends on the condition that the inverse of the agent's marginal utility is unbounded. Phelan shows that in an hidden endowment economy with a risk neutral planner and a continuum of ex ante identical agents, if the agents' marginal utilities are bounded away from zero then in the long run the expected utility of a positive measure of the agents will diverge to its upper limit and the utility of others will diverge to the lower limit.

With two-sided moral hazard, the submartingale characterizations do not pin down the definite directions for the movements of utility and consumption. As discussed in Section 4, each agent's marginal utilities may satisfy either (20) or the reverse inequality, depending upon the output history. In addition, if both agents face the moral hazard constraints then the submartingale conditions imply that the two marginal utility ratios both grow in expectation, which only demonstrates the persistent fluctuations in consumption and utility as shown in Proposition 7 part (a).

Nevertheless, the results in one-sided moral hazard can still be seen as special cases of two-sided moral hazard. Part (a) of Proposition 8 generalizes [18], [41] and [4] to a risk-averse principal, and part (b) generalizes [35] to the case of a finite number of agents. Part (c) is new. It says that if both agents' marginal utilities are bounded away from zero when consumptions go to infinity and are bounded away from ∞ when consumptions go to $-\infty$, then agents' fates in the long run are a priori indeterminate. This is in contrast to one-sided moral hazard in part (a), where agents' fates are completely predetermined by the identity of the agent who faces the incentive constraints.

Finally, consider the situation when default payoff $\underline{U}_i > -\infty$, i.e. agent i has limited commitment to long-term contracting.[17]

[17] For example, a consumer may declare bankruptcy; a sovereign country can repudiate its contractual obligations.

The utility boundary $\underline{U}_i$ is a *reflecting barrier* if with positive probability agent i's lifetime utility is bounced back inside after hitting the boundary, i.e. for any history h^t at which $U_i(\sigma|h^t) = \underline{U}_i$ there is some history (h^t, θ^{t+1}) at which $U_i(\sigma|h^t, \theta^{t+1}) > \underline{U}_i$.

Proposition 9. *Assume A1-A2, A4, and* $\underline{c}_i = 0$, $\forall i$. *If* $\underline{U}_i > -\infty$ *then in every optimal contract with positive probability agent i's expected utility hits its lower boundary* $\underline{U}_i$ *in finite time, and the boundary is a reflecting barrier.*

Proof. See Appendix. □

This result is not very surprising given the submartingale characterization. It is related to the findings of Atkeson and Lucas [5] and Phelan [34]. In particular, [5] studies optimal unemployment insurance when the agent is subject to unobservable employment risks; the agent can not trade all future rights to consumption for current consumption and therefore has limited commitment to long-term contracting. [34] considers a competitive insurance market where agents endure private endowment shocks and can not legally commit not to walk away from a contract that entails a level of utility below what can be obtained from another insurer. In both of these studies, agents' expected utilities are repeatedly driven down to their lower boundaries.

It is perhaps worthwhile pointing out some of the differences between limited commitment with moral hazard and the models of self-enforcing contracts with symmetric information. In the latter models (see Thomas and Worrall [43], [42], Kocherlakota [23], and Ray [36]), if the discount factor is low or if the autarkic outcomes are sufficiently attractive then autarky is the only self-enforcing outcome. However, as shown in Corollary 2 in the current model autarky is never optimal.

Moreover, with symmetric information it has been found that the utility of the agent without commitment grows over time, whence is "back-loaded" (e.g. Thomas and Worrall [43], [42] and Ray [36]). This result to some extent is overturned in the moral hazard environment: an agent with limited commitment repeatedly hits his *lower* utility boundary.

This difference largely comes from the different ways incentives are handled in the two environments. With symmetric information, by back-loading the payments to the agent optimal contracts relax the commitment constraints over time; in the meantime the effort incentives can be provided by the current payments and by the threat of punishment in the future, which however need not be carried out in equilibrium. In contrast, when efforts are hidden optimality requires that the expected utilities of the agents continue to spread out. In particular the submartingale characterization implies that there is always some downward movement in expected utilities, which results in the agent repeatedly hitting his lower utility boundary. Note that in this model (as well as in [34] and [5]) one-period contracts are assumed to be enforceable. This however is not essential for the difference, because in repeated partnership games with hidden action (e.g. Abreu et al. [1]) agents have no commitment to one-period contracts but their expected utilities still may hit the lowest levels with positive probabilities due to the bang-bang nature of the sequential equilibrium strategies.

Finally, to illustrate the characterization results I compute an example using the algorithm developed in Section 3. The example is parameterized as follows. Each agent's outputs take on two values, high = 5 and low = 0.5. For convenience assume that each agent's effort is identified with the probability that the high output will occur: $a = \text{Prob}(\text{output} = 5)$. Each agent has the utility function $c^{1-\sigma}/(1-\sigma) - g(a)$. Assume risk parameter $\sigma = 2$; discount factor $\delta = 0.98$; $g(a) = 50a^2$. In the computation I use polynomials to approximate the value functions.[18] For feasibility in the computation the state variable (agent 1's promised utility) U is restricted to the interval $[-100, -79.09]$ where -100 is the autarkic utility level and $-79.09 = Q_1 - 70$, where $Q_1 = -9.09$ is the supremum on agent 1's lifetime utilities with full commitment.

I find that the optimal value function is monotone decreasing and concave (Figure 1). To illustrate the possible dynamics of the optimal allocations, two sample paths are computed for two different scenarios. In each case, expected utility and effort paths are computed for 400 periods. In the first case, the two agents start with quite different initial expected utilities, but they gradually converge after 400 periods (see Figure 2). In the second scenario, agents initially start with similar expected utilities, but they grow apart over time (see Figure 3).

Along both sample paths, agents' utilities and efforts are negatively correlated, illustrating the moral hazard problem; and they exhibit the phenomena of working-poor and leisure-class. For instance, along the first sample path agent 1 starts out as a member of the leisure class: in the first period his expected utility is above the autarkic level and his effort is below the optimal autarkic level (which is 0.036). Over time, agent 1's expected utility falls and his effort rises. On the other hand, agent 2 starts out as a working-poor with expected utility below the autarkic level and effort above the autarkic level. Over time agent 2's expected utility gradually rises accompanied by falling effort levels. Moreover, after about 200 periods both agents' expected utilities are above the autarkic levels and their efforts are below the autarkic levels, showing that both risk sharing and moral hazard are at work.

[18] See Ljungqvist and Sargent [28] Chapter 3 for an introduction to numerically solving dynamic programming problems using polynomial approximations.

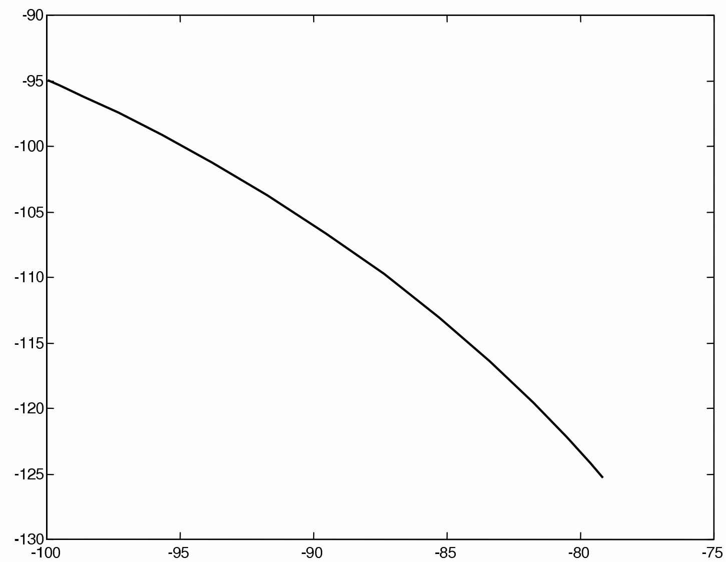

Figure 1: Value function V. x-axis: expected utility of agent 1.

6 Conclusion

This paper has studied optimal risk-sharing in a dynamic model with two-sided hidden efforts. The contracting problem admits a recursive formulation, and the optimal value function is the unique fixed point of a contraction mapping operator despite the fact that the value function also appears in the constraints. The recursive formulation and the algorithm should find use in other applications of the model. With additional assumptions on preferences and technologies optimal contracts are recursively optimal: every continuation contract of an optimal contract is itself optimal. Moreover, the marginal utility ratio between one agent and another is a submartingale, which is a testable restriction against the alternative assumption of one-sided moral hazard. The characterization results imply that it is in general important to restrict an agent whose moral hazard constraint is binding from saving through another agent within the risk-sharing group.

This paper contributes to the dynamic insurance literature by considering multi-sided hidden efforts as the main impediment to efficient risk sharing as opposed to hidden information or one-sided moral hazard. It remains to be seen whether the method and results of this paper can be usefully extended to risk sharing problems with capital accumulation.

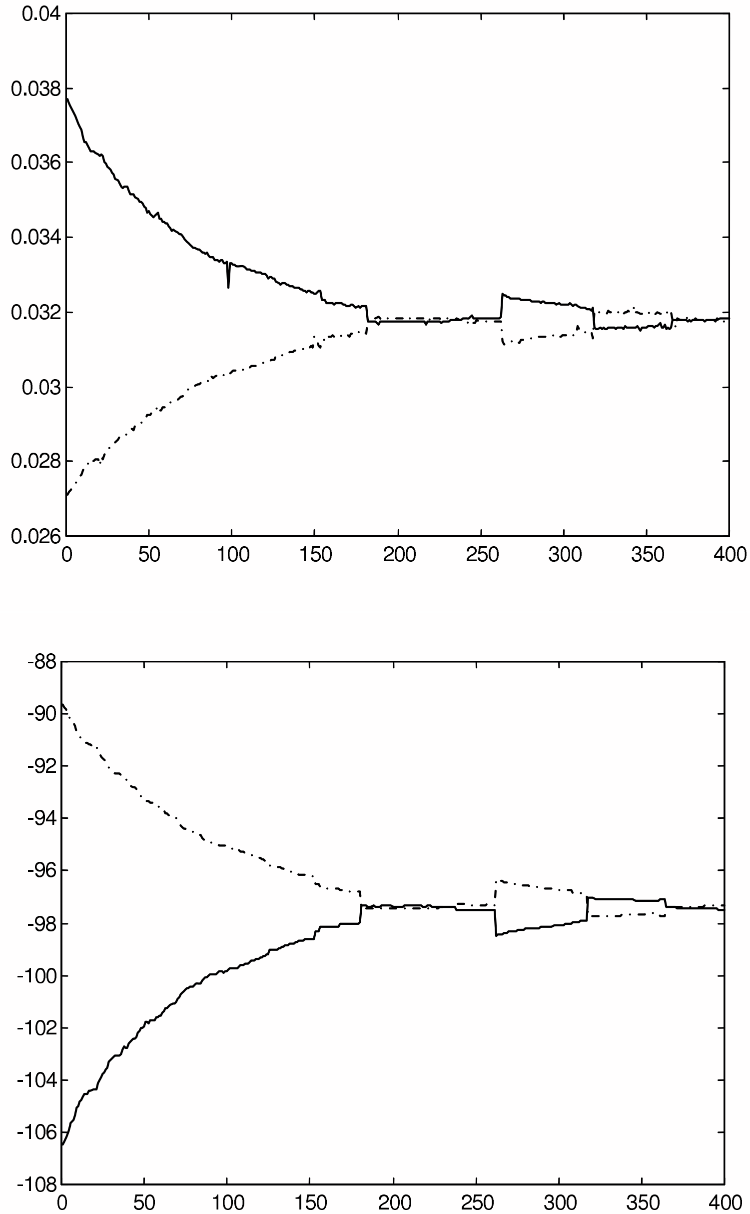

Figure 2: Sample path 1. Upper panel: effort paths. Lower panel: expected utility paths. '-.-.-' = agent 1; '—' = agent 2.

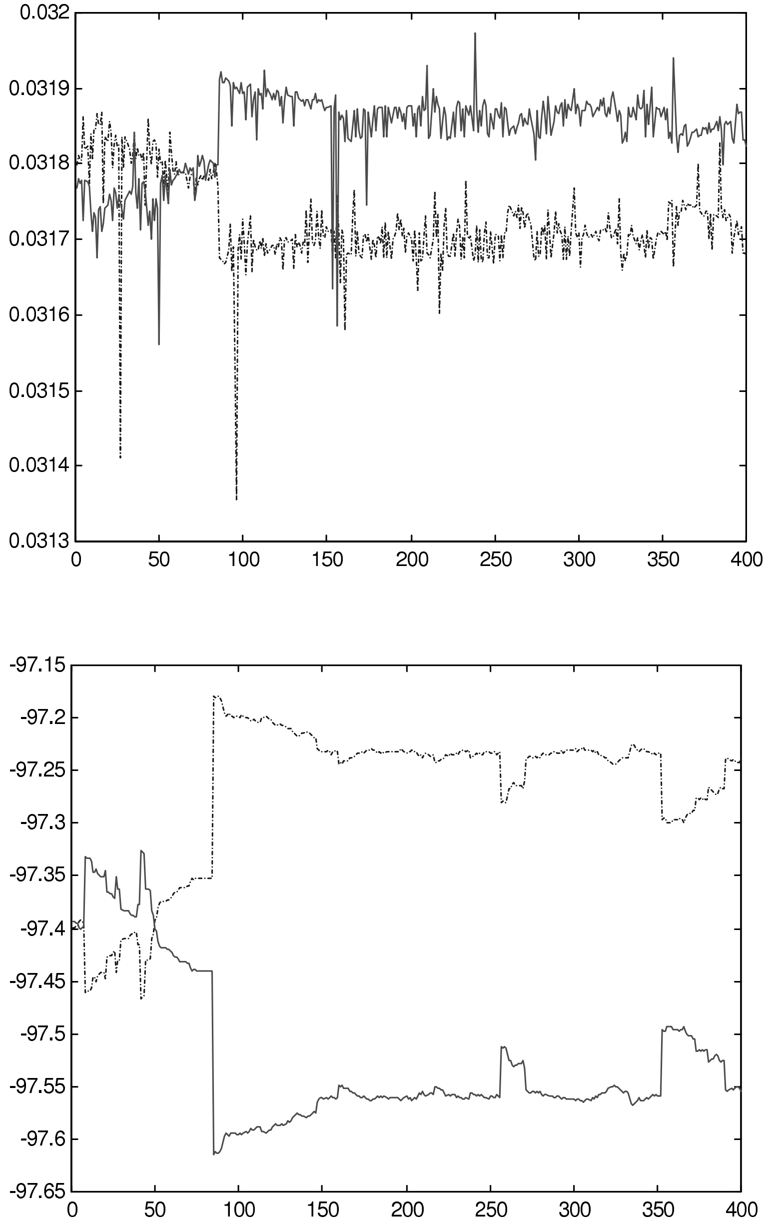

Figure 3: Sample path 2. Upper panel: effort paths. Lower panel: expected utility paths. '-.-.-' = agent 1; '—' = agent 2.

A Proofs

Proof of Lemma 1. To prove part (a), let $(\xi^n) \to \xi$ and $(y^n) \to y$ with $\xi^n \in \mathcal{D}$ and $y^n \in \mathcal{V}(\xi^n)$ for all n. Then there exists a sequence of incentive feasible contracts $(\sigma^1, ..., \sigma^n, ...)$ with $U_1(\sigma^n) = \xi^n$ and $U_2(\sigma^n) = y^n$.

I shall construct a contract σ^* with $U_1(\sigma^*) = \xi$ and $U_2(\sigma^*) = y$. For this purpose, I first show that the sequence of consumptions $(c_i^n(h^t))$, prescribed by contracts (σ^n) to agent i at each history $h^t \neq h^0$, is contained in some compact interval.

By resource constraint (1), for all $h^t \neq h^0$, $u_j(c_j^n(h^t)) \leqslant u_j(\theta_1 + \theta_2 - c_i^n(h^t)) \leqslant u_j(\bar{\theta} - c_i^n(h^t))$, where $\bar{\theta} \equiv \max_\theta(\theta_1 + \theta_2)$ is the maximum total output.

Since the strictly concave function $f(z) \equiv u_j(\bar{\theta} - u_i^{-1}(z))$ is bounded from above by some affine functions $f(z) \leq \alpha_1 z + \beta_1$, $f(z) \leq \alpha_2 z + \beta_2$ with $\alpha_1 < \alpha_2 < 0$, for all history $\hat{h}^t$ we have

$$\begin{aligned}
& U_j(\sigma^n) + G_j^n \\
= \;& E \sum_{h^t} \delta^{t-1} u_j(c_j^n(h^t)) \\
\leq \;& E \sum_{h^t} \delta^{t-1} f(u_i(c_i^n(h^t))) \\
\leq \;& E \sum_{h^t \neq \hat{h}^t} \delta^{t-1} \left(\alpha_1 u_i(c_i^n(h^t)) + \beta_1\right) + \delta^{t-1} p(\hat{h}^t) \Big(\alpha_2 u_i(c_i^n(\hat{h}^t)) + \beta_2\Big) \\
\leq \;& E \sum_{h^t} \delta^{t-1} \left(\alpha_1 u_i(c_i^n(h^t)) + \beta_1\right) + \delta^{t-1} p(\hat{h}^t) \Big((\alpha_2 - \alpha_1) u_i(c_i^n(\hat{h}^t)) + \beta_2 - \beta_1\Big) \\
\leq \;& \alpha_1 \left(U_i(\sigma^n) + G_i^n\right) + \beta_1 + \delta^{t-1} p(\hat{h}^t) \Big((\alpha_2 - \alpha_1) u_i(c_i^n(\hat{h}^t)) + \beta_2 - \beta_1\Big) \qquad (21)
\end{aligned}$$

where $G_j^n \equiv E \sum_{h^t} \delta^{t-1} g_j(s_j^n(h^t))$ is the discounted expected disutility of agent j, and $p(\hat{h}^t)$ is the ex ante probability that history $\hat{h}^t$ will be reached.

By line (21), $u_i(c_i^n(\hat{h}^t)) \to -\infty$ implies $U_j(\sigma^n) \to -\infty$. Reversing the roles of (α_1, β_1) and (α_2, β_2), one shows that $u_i(c_i^n(\hat{h}^t)) \to +\infty$ also implies $U_j(\sigma^n) \to -\infty$. Neither is possible.

Therefore given resource constraint (1), the sequence of consumptions $(c_i^n(h^t))$ is contained in some interval $[\underline{b}_i(h^t), \overline{b}_i(h^t)]$, for $i = 1, 2$ and for all $h^t \neq h^0$.

Now I construct the desired contract σ^*. It is useful to decompose each agent i's payoff from contract $\sigma^n = (c_i^n, s_i^n)$ as follows:

$$U_i(\sigma^n) = E_{h^1}[u_i(c_i^n(h^1)) + \delta U_i(\sigma^n | h^1)] - g_i(s_i^n(h^0)).$$

Since the sequence $((c_i^n(h^1)), s_i^n(h^0))$ is contained in a compact set, it has a convergent subsequence. Re-index the convergent subsequence as $(c_i^m(h^1), s_i^m(h^0))$, and denote the limit by $((c_i^*(h^1)), s_i^*(h^0))$. Since $U_2(\sigma^m) = y^m \to y$ and $U_1(\sigma^m) = \xi^m \to \xi$, it follows that for all h^1, $U_i(\sigma^m|h^1)$ converges to some limit $U_i^*(h^1)$ with

$$E_{h^1}[u_1(c_1^*(h^1)) + \delta U_1^*(h^1)] - g_1(s_1^*(h^0)) = \xi$$

$$E_{h^1}[u_2(c_2^*(h^1)) + \delta U_2^*(h^1)] - g_2(s_2^*(h^0)) = y.$$

Clearly, $(c_i^*(h^1))$ and $(s_i^*(h^0))$ satisfy the one-step incentive compatibility constraints with continuation payoffs $(U_i^*(h^1))$.[19]

Consider next any history h^1. By similar argument, we can find a subsequence $(\sigma^q|h^1)$ of continuation contracts $(\sigma^m|h^1)$ such that contingent consumptions at each h^2 converge to some $(c_i^*(h^2))$, actions at h^1 converge to some $(s_i^*(h^1))$, and $U_i(\sigma^q|h^2)$ converges to some $U_i^*(h^2)$. Moreover, the tuple $((c_i^*(h^2)), s_i^*(h^1), (U_i^*(h^2)))$ satisfy the one-step incentive constraints and attain expected utility $U_i^*(h^1)$. Continuing the process ad infinitum, we obtain the new contract σ^*. By construction, σ^* is incentive feasible, $U_1(\sigma^*) = \xi$ and $U_2(\sigma^*) = y$.

Finally, note that by choosing $\alpha_2 = \alpha_1 < 0, \beta_2 = \beta_1$, equation (21) implies

$$U_j(\sigma) + G_j \le \alpha_1(U_i(\sigma) + G_i) + \beta_1.$$

Since G_i is bounded for each i, this proves part (b). $\square$

Proof of Lemma 4. Consider first $\underline{c}_2 = -\infty$, in which case

$$Q = (u_1(\infty) - g(\underline{a}_1))/(1-\delta).$$

Let $q \equiv (u_1(\underline{c}_1) - g(\underline{a}_1))/(1-\delta)$. If A2 is satisfied then $q = -\infty$ and for every $\xi \in (q,\ Q)$, there is some c_1 such that $(u_1(c_1) - g(\underline{a}_1))/(1-\delta) = \xi$; if A2' is satisfied then $q = 0$ and for every $\xi \in [q,\ Q)$, there is some c_1 such that $(u_1(c_1) - g(\underline{a}_1))/(1-\delta) = \xi$. In either case, letting $c_2(\theta) = \theta_1 + \theta_2 - c_1\ \forall\theta$ and a_2 be any action implementable by such $c_2(\theta)$, we obtain a stationary incentive feasible contract that attains ξ for agent 1.

Consider next $\underline{c}_2 = 0$, in which case Q is defined by (5). First, I show that every incentive feasible contract delivers to agent 1 some expected utility less than Q. By the definition of Q, at each history h^t the contemporaneous expected utility of agent 1 satisfies

$$\begin{aligned}
&\sum_\theta p_1(\theta_1|s_1(h^t))p_2(\theta_2|s_2(h^t))u_1(c_1(h^t,\theta)) - g_1(s_1(h^t)) \\
<\ & (1-\delta)Q(s_2(h^t)) \\
\le\ & (1-\delta)Q.
\end{aligned}$$

[19] This is to say Nash correspondence has a closed graph.

It only remains to show that every ξ in $(-\infty, Q)$ (under assumption A2) or in $[0, Q)$ (under A2') can be attained by some incentive feasible contract.

By the definition of Q, for every $\epsilon > 0$ there exists some a_2^* such that $Q > Q(a_2^*, 1) > Q - \epsilon$.

By the Theorem of the Maximum, $Q(\lambda, a_2^*)$ is continuous in λ. Therefore there exists $\lambda^* < 1$ such that $\xi^* = Q(\lambda^*, a_2^*) > Q - \epsilon$. Moreover, if assumption A2' holds, then $\lambda \in [0, 1]$ and $Q(0, a_2^*) = 0$, and by continuity function $Q(\lambda, a_2^*)$ attains all values in interval $[0,\ \xi^*]$. If assumption A2 holds, then $\lambda \in (-\infty, 1]$ (if $\underline{c}_1 = -\infty$) or $\lambda \in (0, 1]$ (if $\underline{c}_1 = 0$), and in either case by continuity function $Q(\lambda, a_2^*)$ attains all values in $(u_1(\underline{c}_1) = -\infty,\ \xi^*]$.

To complete the proof, I only need to show that for every $\lambda < 1$ as long as $\xi = Q(\lambda\,, a_2^*)$ is defined then there is an incentive feasible contract that delivers lifetime utility ξ to agent 1. Let $(a_1^*(\lambda), c_2^*(\cdot))$ be an optimal solution to Program $Q(\lambda\,, a_2^*)$. Then the consumption plan defined by $c_2(\theta) \equiv u_2^{-1}[u_2(c_2^*(\theta_2)) - w]$, where $w > 0$ is sufficiently large so that $c_2(\theta) \leq \min_\theta (1 - \lambda)(\theta_1 + \theta_2)$, also implements action a_2^*. Letting agent 1's consumption $c_1(\theta) = \lambda(\theta_1 + \theta_2)$, $\forall\theta$, we obtain a desired stationary incentive feasible contract. $\square$

Proof of Lemma 5. Let σ be a contract satisfying the hypothesis in the proposition. If $U_2(\sigma|\theta) < V(U_1(\sigma|\theta))$ then replace $\sigma|\theta$ with some incentive feasible contract $\sigma'|\theta$ such that $U_1(\sigma'|\theta) = U_1(\sigma|\theta)$ and $U_2(\sigma'|\theta) = V(U_1(\sigma'|\theta))$ (the existence of such a $\sigma'|\theta$ is guaranteed by Lemma 2). Then reduce agent 2's consumption $c_2(\theta)$ by an appropriate amount $\epsilon(\theta)$ so that the total expected utility $u_2(c_2(\theta) - \epsilon(\theta)) + \delta U_2(\sigma'|\theta)$ is equal to the original level $u_2(c_2(\theta)) + \delta U_2(\sigma|\theta)$. The result is a desired contract σ'. $\square$

Proof of Lemma 6. By Lemma 2, for every $\xi \in \mathcal{D}$ there is an optimal contract σ with $U_1(\sigma) = \xi$ and $U_2(\sigma) = V(\xi)$, and by Lemma 5 the continuation contracts $(\sigma|\theta)$ can be chosen so that $U_2(\sigma|\theta) = V(U_1(\sigma|\theta))$. This implies $TV(\xi) \geq V(\xi)$. On the other hand, given $\xi \in \mathcal{D}$, let $(a_i, (c_i(\theta)), (U(\theta)))$ be a policy vector that satisfies the constraints of (FE) and attains some value v in the objective function. Then there is an incentive feasible contract $\sigma = (a_i, (c_i(\theta)), (\sigma|\theta))$, where for each θ contract $\sigma|\theta$ attains payoffs $(U(\theta), V(U(\theta)))$, such that $U_1(\sigma) = \xi$ and $U_2(\sigma) = v$. Therefore $TV(\xi) \leq V(\xi)$. Hence V is a fixed point of T. The existence of value function V (Lemma 2) guarantees that the sup in (FE) is attained at this fixed point. $\square$

Proof of Proposition 2. First I show that $Tf \in B(S)$. Let $[\underline{\xi},\ \overline{\xi}\,] \subset \mathcal{D}_\infty$ and $[\underline{\xi},\ \overline{\xi}\,] \supset S$.

By equation (7), given a feasible policy vector $(a_i, c_i(\cdot), U(\cdot))$, agent 1's current expected utility is bounded below by either $K \equiv \underline{\xi} - \delta\overline{\xi}$ or $K = 0$ (if A2' holds):

$$\sum_{\theta} p(\theta_1|a_1)p(\theta_2|a_2)u_1(c_1(\theta)) \geqslant K.$$

Since function $f(z) \equiv u_2(\bar{\theta} - u_1^{-1}(z))$, where $\bar{\theta} = \max_\theta(\theta_1 + \theta_2)$, is strictly decreasing and strictly concave, it follows that

$$\begin{aligned}\sum_{\theta} p(\theta_1|a_1)p(\theta_2|a_2)u_2(c_2(\theta)) &\leq \sum_{\theta} p(\theta_1|a_1)p(\theta_2|a_2)f(u_1(c_1(\theta))) \\ &\leq f\left(\sum_{\theta} p(\theta_1|a_1)p(\theta_2|a_2)u_1(c_1(\theta))\right) \\ &\leq f(K).\end{aligned}$$

Therefore Tf is bounded from above by $f(K) - g(\underline{a}_2) + \delta \sup f$.

On the other hand, by the definitions of Q and $Q(\lambda, a_2)$ in (5) and (4), there is some $a_2^* \in A_2$ such that $Q > Q(1, a_2^*) > \overline{\xi}$. By the Theorem of the Maximum, $Q(\lambda, a_2^*)$ is continuous in λ. Therefore, there exists $\lambda^* < 1$ such that $Q(\lambda^*, a_2^*) = \overline{\xi}$ and for every $\xi \in [\underline{\xi},\, \overline{\xi}] \supset S$ there exists $\lambda_\xi < \lambda^*$ with $Q(\lambda_\xi, a_2^*) = \xi$.

Let $(c_2(\theta))$ be a consumption vector that implements a_2^*. Then for large $w > 0$ the consumption vector $c_2^*(\theta) \equiv u_2^{-1}[u_2(c_2(\theta_2)) - w] \leq \min_\theta(1 - \lambda^*)(\theta_1 + \theta_2)$ also implements a_2^*. For all $\xi \in S$, let $a_1^*(\lambda_\xi)$ be an optimal solution to Program (4). Then the policy vector $(a_1^*(\lambda_\xi),\, a_2^*,\, c_1(\theta) = \lambda_\xi(\theta_1 + \theta_2),\, c_2^*(\theta),\, U(\theta) = \xi)$ satisfies the constraints of (FE) for promised utility $\xi \in S$, and therefore

$$\forall \xi \in S: \quad Tf(\xi) \geqslant \sum_{\theta_2} p(\theta_2|a_2^*)u_2(c_2^*(\theta_2)) - g(a_2^*) - \delta||f||.$$

Hence $Tf \in B(S)$.

By Blackwell's sufficiency theorem for a contraction mapping,[20] one only need prove that operator T satisfies monotonicity and discounting. Discounting can be easily seen.

To prove monotonicity, let $f, g \in B(S)$ with $f \leq g$.

Fix $\xi \in S$. Given function f, suppose $(a_i, c_i(\cdot), U(\cdot))$ satisfies the constraints in program (FE) and let ϕ_f be the corresponding value of the objective function. I shall show that there exists a policy vector $(\hat{a}_i, \hat{c}_i(\cdot), \hat{U}(\cdot))$ that satisfies the constraints of program (FE) when f is replaced by g, and the corresponding value of the objective function is at least as great as ϕ_f. It then would follow that $Tf(\xi) \leq Tg(\xi)$ and the proof is complete.

[20]See Stokey, Lucas, with Prescott [40], Theorem 3.3 on page 54.

The desired new policy is constructed as follows. First, replace f with g in program (FE). With the initial policy $(a_i, c_i(\cdot), U(\cdot))$, the only constraint that could be violated is (9).

Then for each θ with $g(U(\theta)) > f(U(\theta))$, reduce $c_2(\theta)$ by an appropriate amount so that agent 2's expected utility at state θ remains unchanged:

$$u_2(\hat{c}_2(\theta)) + \delta g(U(\theta)) = u_2(c_2(\theta)) + \delta f(U(\theta)).$$

Clearly, with the new policy $(a_i, \hat{c}_i(\cdot), U(\cdot))$ all of the constraints are satisfied and the value of the objective function exactly equals ϕ_f. $\square$

Lemma 11 below, which parallels Lemma 1, is used to prove Lemma 7.

For $f \in B^u(S)$ and $\xi \in S$, define set $F(\xi) \subseteq \Re$ as follows: $y \in F(\xi)$ if and only if there exists policy vector $\rho \equiv (a_i, c_i(\cdot), U(\cdot))$ satisfying (7) through (10) and

$$y = U_2(\rho) \equiv \sum_{\theta_1} \sum_{\theta_2} p(\theta_1|a_1)p(\theta_2|a_2)\Big[u_2(c_2(\theta)) + \delta f(U(\theta))\Big] - g_2(a_2).$$

Note that $F(\xi)$ is nonempty for all $\xi \in S$.[21]

Lemma 11. *If A1 and either A2 or A2' are satisfied, then* (a) *the correspondence $F : S \twoheadrightarrow \Re$ has a closed graph, i.e. for any two sequences $(\xi^n) \to \xi \in \Re$ and $(y^n) \to y \in \Re$ with $\xi^n \in S$ and $y^n \in F(\xi^n)$ for all n, we have $\xi \in S$ and $y \in F(\xi)$;* (b) *the image $F(S)$ is bounded from above.*

Proof of Lemma 11. The proof parallels that of Lemma 1. I will first show F has a closed graph; boundedness will follow in the process. Let $(\xi^n) \to \xi$ and $(y^n) \to y$ be as specified. Therefore there exists a sequence of incentive feasible policies $(\rho^1, ..., \rho^n, ...)$ with $U_1(\rho^n) = \xi^n$ and $U_2(\rho^n) = y^n$, where $U_1(\rho) \equiv \sum_\theta p(\theta_1|a_1)p(\theta_2|a_2)\big[u_1(c_1(\theta)) + \delta U(\theta)\big] - g_1(a_1)$.

To show that $y \in F(\xi)$, I shall construct a policy ρ^* with $U_1(\rho^*) = \xi$ and $U_2(\rho^*) = y$. For this purpose, I will show first that the sequence of consumptions $(c_i^n(\theta))$, prescribed by policies (ρ^n) at any θ, is contained in some compact interval.

By resource constraint (1), for all θ, $u_j(c_j^n(\theta)) \leqslant u_j(\theta_1 + \theta_2 - c_i^n(\theta)) \leqslant u_j(\bar{\theta} - c_i^n(\theta))$, where $\bar{\theta} \equiv \max_\theta(\theta_1 + \theta_2)$ is the maximum total output.

Since the strictly concave function $v(z) \equiv u_j(\bar{\theta} - u_i^{-1}(z))$ is bounded from above by two affine functions, i.e. $v(z) \leq \alpha_1 z + \beta_1$, $v(z) \leq \alpha_2 z + \beta_2$ for some α_i, β_i

[21] Let $U(\theta) = \xi, \forall\theta$. Then by the argument of Lemma 4, there exists one-period contract $(a_i, c_i(\cdot))$ that satisfies constraints (7) through (10).

with $\alpha_2 > \alpha_1$, we have for an arbitrary $\hat{\theta}$,

$$\begin{aligned} E_\theta[u_j(c_j^n(\theta))] &\leqslant E_\theta[v(u_i(c_i^n(\theta)))] \\ &\leqslant \sum_{\theta \neq \hat{\theta}} p(\theta|a^n)[\alpha_1 u_i(c_i^n(\theta)) + \beta_1] + p(\hat{\theta}|a^n)[\alpha_2 u_i(c_i^n(\hat{\theta})) + \beta_2] \\ &\leqslant \sum_{\theta} p(\theta|a^n)[\alpha_1 u_i(c_i^n(\theta)) + \beta_1] + p(\hat{\theta}|a^n)[(\alpha_2 - \alpha_1) u_i(c_i^n(\hat{\theta})) + \beta_2 - \beta_1] \\ &\leqslant \alpha_1 E_\theta[u_i(c_i^n(\theta))] + \beta_1 + p(\hat{\theta}|a^n)[(\alpha_2 - \alpha_1) u_i(c_i^n(\hat{\theta})) + \beta_2 - \beta_1] \qquad (22) \end{aligned}$$

By the last line (22), we have $u_i(c_i^n(\hat{\theta})) \to -\infty$ implies $U_j(\rho^n) \to -\infty$. Reversing the roles of (α_1, β_1) and (α_2, β_2), one shows that $u_i(c_i^n(\hat{\theta})) \to +\infty$ also implies $U_j(\rho^n) \to -\infty$. Neither is possible.

Therefore the sequence of consumptions $(c_i^n(\theta))$ is contained in some closed interval $[\underline{b}_i(\theta), \overline{b}_i(\theta)]$, for $i = 1, 2$ and for all θ.

Now since the sequence $(c_i^n(\cdot), a_i^n)$ is contained within a compact set, it has a convergent subsequence, re-indexed by m, with limit $(c_i^*(\cdot), a_i^*)$. Moreover, since $U_2(\rho^m) = y^m \to y$ and $U_1(\rho^m) = \xi^m \to \xi$, it follows that for all θ, $U^m(\theta)$ converges to some limit $U^*(\theta) \in S$ with

$$E_\theta[u_1(c_1^*(\theta)) + \delta U^*(\theta)] - g_1(a_1^*) = \xi$$

and $f(U^m(\theta))$ converges to some limit $f^*(\theta)$ with

$$E_\theta[u_2(c_2^*(\theta)) + \delta f^*(\theta)] - g_2(a_2^*) = y.$$

Since f is upper semi-continuous, it follows $f(U^*(\theta)) \geqslant f^*(\theta)$.

Clearly, $(c_i^*(\theta))$, (a_i^*), $U^*(\theta)$, and $f^*(\theta)$ satisfy constraints (7) through (10). For all θ if $f(U^*(\theta)) > f^*(\theta)$ then reduce $c_2^*(\theta)$ by some $\epsilon(\theta)$ so that $u_2(c_2^*(\theta)) + \delta f^*(\theta) = u_2(c_2^*(\theta) - \epsilon(\theta)) + \delta f(U^*(\theta))$. Then $(c_1^*(\theta), c_2^*(\theta) - \epsilon(\theta), a_i^*, U^*(\theta))$ is a desired policy vector.

Finally, by choosing $\alpha_2 = \alpha_1, \beta_2 = \beta_1$, equation (22) implies for any policy $\rho = (c_i(\cdot), a_i, U(\cdot))$,

$$E_\theta[u_2(c_2(\theta))] \leqslant \alpha_1 E_\theta[u_1(c_1(\theta))] + \beta_1.$$

Since f and disutility function g_2 are all bounded, part (b) is proven. $\square$

Proof of Lemma 7. Part (a) follows because by Lemma 11 in the Appendix, for $\xi \in S$ set $F(\xi)$ is closed and bounded from above and $Tf(\xi) = \max F(\xi)$.

Let $(\xi^n) \to \xi \in S$ with $\xi^n \in S$ and $y^n = Tf(\xi^n)$ for all n. By Lemma 11, the sequence (y^n) is bounded from above and $\limsup(y^n) \in F(\xi)$. Therefore $Tf(\xi) \geq \limsup(y^n)$, showing Tf is upper semi-continuous. $\square$

Proof of Proposition 3. It is clear that $S_1 \subseteq S_0$ is closed hence compact. Since f_1 is generated by the restriction of f_0 to S_1 but f_0 is generated by f_0 ($f_0 = Tf_0$), then $f_1(\xi) \leqslant f_0(\xi)$ for all $\xi \in S_1$. Now suppose for $k \geqslant 1$, $S_k \subseteq S_{k-1}$ is compact and $f_k(\xi) \leqslant f_{k-1}(\xi)$, $\forall \xi \in S_k$. Then $S_{k+1} \subseteq S_k$ is closed hence compact. Since f_{k+1} is generated by the restriction of f_k to S_{k+1}, f_k is generated by f_{k-1} restricted to $S_k \supseteq S_{k+1}$, and $f_k \leqslant f_{k-1}$ on S_k, it follows that for all $\xi \in S_{k+1}$, $f_{k+1}(\xi) \leqslant f_k(\xi)$. Therefore (S_k) is a nested decreasing sequence of compact subsets of $\Re$ so it must converge to some nonempty compact set $\hat{S}$, and for all $\xi \in \hat{S}$, the decreasing sequence $(f_k(\xi))$ must converge to some limit $f(\xi)$. Restricted to domain $\hat{S}$, the sequence (f_k) converges to f uniformly (i.e. in sup metric), therefore f is upper semi-continuous since $f_k \in B^u(\hat{S})$ for all k and $B^u(\hat{S})$ is closed.

Let $\hat{f}$ be the unique fixed point of operator T defined by (FE) on space $B(\hat{S})$. I shall show $f = \hat{f}$. For a given $\xi \in \hat{S}$, let $\rho^k = (c_i^k(\theta), a_i^k, U^k(\theta))$ be a policy vector that attains $f_k(\xi)$. By similar argument as in the proof of Lemma 11, the sequence (ρ^k) is contained in some compact subset of a Euclidean space therefore has a convergent subsequence, re-indexed as (ρ^m), with limit $\rho^* = (c_i^*(\theta), a_i^*, U^*(\theta))$.

Moreover, since $U_2(\rho^m) = f_m(\xi) \to f(\xi)$ and $U_1(\rho^m) = \xi$, $\forall m$, it follows that

$$E_\theta[u_1(c_1^*(\theta)) + \delta U^*(\theta)] - g_1(a_1^*) = \xi$$

and for all θ, $f(U^m(\theta))$ converges to some limit $f^*(\theta)$ with

$$E_\theta[u_2(c_2^*(\theta)) + \delta f^*(\theta)] - g_2(a_2^*) = f(\xi).$$

Since f is upper semi-continuous, it follows $f(U^*(\theta)) \geqslant f^*(\theta)$.

Clearly, $(c_i^*(\theta))$, (a_i^*), $U^*(\theta)$, and $f^*(\theta)$ satisfy constraints (7) through (10) when $f^*(\theta)$ is in place of $f(U^*(\theta))$ in (9). For all θ with $f(U^*(\theta)) > f^*(\theta)$, reduce $c_2^*(\theta)$ by some $\epsilon(\theta)$ so that $u_2(c_2^*(\theta)) + \delta f^*(\theta) = u_2(c_2^*(\theta) - \epsilon(\theta)) + \delta f(U^*(\theta))$. Now $(c_1^*(\theta), c_2^*(\theta) - \epsilon(\theta), a_i^*, U^*(\theta))$ is a policy vector that attains $f(\xi)$ given f as the continuation function. It follows that $Tf(\xi) \geq f(\xi)$. Hence $T^k f \geq T^{k-1} \cdots \geq f$ for all $k \geq 1$. Therefore $\hat{f} \geq f$ as $T^k f \to \hat{f}$.

On the other hand, let $\hat{f}_k$ be the restriction of f_k to $\hat{S}$. Then for all $\xi \in \hat{S}$, $T\hat{f}_k(\xi) \leqslant f_{k+1}(\xi) = \hat{f}_{k+1}(\xi)$. Therefore $T^k \hat{f}_0 \leqslant T^{k-1}\hat{f}_1 \cdots \leqslant \hat{f}_k$, $\forall k \geq 1$. Since $T^k \hat{f}_0 \to \hat{f}$ and $\hat{f}_k \to f$, then $\hat{f} \leq f$.

It only remains to show $\hat{S} = \mathcal{D}$. For every $\xi \in \hat{S}$, one can construct in the standard way an incentive compatible contract σ, with $U_1(\sigma) = \xi$, that satisfies the resource constraints and the limited commitment constraints $U_i(\sigma) \geqslant \underline{U}_i$: there exists a policy vector $(c_i(\theta), a_i, U(\theta))$ that generates payoff pair $(\xi, f(\xi))$; the tuple $(c_i(\theta), a_i)$ will be the first-period component of σ. Then the policy vector that generates each $U(\theta)$ will provide the next-period components of σ conditional on history θ. Continuing with the process indefinitely, we obtain contract σ.

Therefore $\hat{S} \subseteq \mathcal{D}$. On the other hand, since $\mathcal{D} \subseteq S_0$, then $f_0 \geqslant V \geqslant \underline{U}_2$ on set $\mathcal{D}$ hence $\mathcal{D} \subseteq S_1$. Since f_1 is generated by f_0 restricted to S_1 and V is generated by V defined on $\mathcal{D}$, it follows that $f_1 \geqslant V \geqslant \underline{U}_2$ on $\mathcal{D}$ therefore $\mathcal{D} \subseteq S_2$. Continuing the process ad infinitum, we have $\mathcal{D} \subseteq S_k$ for all k. Therefore $\mathcal{D} \subseteq \hat{S}$, the intersection of all S_k. It follows $\hat{S} = \mathcal{D}$. $\square$

Proof of Lemma 8. Suppose to the contrary there exist $\theta_i, \theta_i', \theta_j, \theta_j'$ with $\frac{u_i'(\theta_i,\theta_j)}{u_j'(\theta_i,\theta_j)} > \frac{u_i'(\theta_i',\theta_j)}{u_j'(\theta_i',\theta_j)}$ and $\frac{u_i'(\theta_i,\theta_j')}{u_j'(\theta_i,\theta_j')} < \frac{u_i'(\theta_i',\theta_j')}{u_j'(\theta_i',\theta_j')}$. The idea is to construct a Pareto superior new contract by reshuffling agents' consumption/utility across states, which would be a contradiction to optimality.

Denote by $z_i(\theta_i, \theta_j)$ the utility agent i receives at state (θ_i, θ_j) and by $z_i(\theta_i)$ the expected utility agent i receives conditional on output θ_i.

In the first round of reshuffling, choose small $\epsilon > 0$ and $\epsilon' > 0$ as the utility transfers between the two agents so that the resulted utilities are given as follows:

$$\tilde{z}_i(\theta_i, \theta_j) = z_i(\theta_i, \theta_j) + \epsilon$$

$$\tilde{z}_i(\theta_i', \theta_j) = z_i(\theta_i', \theta_j) - \epsilon$$

$$\tilde{z}_i(\theta_i, \theta_j') = z_i(\theta_i, \theta_j') - \epsilon$$

$$\tilde{z}_i(\theta_i', \theta_j') = z_i(\theta_i', \theta_j') + \epsilon$$

where $\epsilon > 0$ and $\epsilon' > 0$ are chosen such that

$$p_j(\theta_j)\epsilon = p_j(\theta_j')\epsilon'.$$

The effects of this round of reshuffling are two: First, because at least one of $u_i(\cdot)$ is strictly concave, the above procedure leads to increased payoffs for agent j conditional on θ_j and θ_j', i.e. there exist $\eta > 0$ and $\eta' > 0$ such that

$$\tilde{z}_j(\theta_j) = z_j(\theta_j) + \eta$$

$$\tilde{z}_j(\theta_j') = z_j(\theta_j') + \eta'.$$

To see this, note that agent j's utility as a function of agent i's utility z_i is given by

$$w_j(z_i) \equiv u_j(\theta_i + \theta_j - u_i^{-1}(z_i)),$$

which is strictly concave; its derivative is given by

$$w_j'(z_i) = -u_j'(c_j^*(\theta))/u_i'(c_i^*(\theta)).$$

The second effect is that agent i's expected payoffs are unchanged conditional on his own output levels θ_i and θ_i'. To see this, note that because $p_j(\theta_j)\epsilon = p_j(\theta_j')\epsilon'$, it follows that

$$\tilde{z}_i(\theta_i) = z_i(\theta_i)$$
$$\tilde{z}_i(\theta_i') = z_i(\theta_i').$$

So agent i's incentive is unaffected.

To smooth out agent j's incentive, reshuffle agents' consumption/utility one more round as follows. For θ_j, and for all $\hat{\theta}_i \in \Theta_i$, increase agent i's payoff by same amount φ:

$$\tilde{\tilde{z}}_i(\hat{\theta}_i, \theta_j) = \tilde{z}_i(\hat{\theta}_i, \theta_j) + \varphi$$

where $\varphi > 0$ is chosen such that

$$\tilde{\tilde{z}}_j(\theta_j) = z_j(\theta_j).$$

Similarly, for θ_j', for all $\hat{\theta}_i \in \Theta_i$, increase agent i's payoff by an equal amount φ':

$$\tilde{\tilde{z}}_i(\hat{\theta}_i, \theta_j') = \tilde{z}_i(\hat{\theta}_i, \theta_j') + \varphi'$$

where $\varphi' > 0$ is chosen such that

$$\tilde{\tilde{z}}_j(\theta_j') = z_j(\theta_j').$$

Clearly, after this round of reshuffling, agents' incentives are not affected from the current period onwards and agent j's continuation payoff remains the same as under the initial contract, but agent i's continuation payoff is increased (by $p_j(\theta_j)\varphi + p_j(\theta_j')\varphi' > 0$). This contradicts the optimality of the continuation contract starting from the current period. □

Proof of Lemma 9. First, by assumption A2 the first constraint of (FE) can be written as weak inequality with the left-hand side $\geqslant$ right-hand side. This is valid because by Lemma 3 the constraint must hold with equality in optimal solutions. For given promised utility $\hat{U}$, let $(a_1^*, a_2^*, c_1^*(\cdot), U^*(\cdot))$ be an optimal solution to program (FE). Then given the vector $(a_1^*, a_2^*, U^*(\cdot))$, $c_1^*(\cdot)$ maximizes the objective function subject to the first three sets of constraints. It follows that there are multipliers $\lambda \geq 0$, $\mu_{a_1} \geq 0$ and $\mu_{a_2} \geq 0$, such that the first-order necessary condition for $c_1^*(\theta_1, \theta_2)$ is given by:

$$\begin{aligned}
&p(\theta_1|a_1^*)p(\theta_2|a_2^*)\left[-u_2'(c_2^*(\theta)) + \lambda u_1'(c_1^*(\theta))\right] \\
&+ \sum_{a_1} \mu_{a_1} u_1'(c_1^*(\theta)) \left[p(\theta_1|a_1^*)p(\theta_2|a_2^*) - p(\theta_1|a_1)p(\theta_2|a_2^*)\right] \\
&+ \sum_{a_2} \mu_{a_2} \left(-u_2'(c_2^*(\theta))\right) \left[p(\theta_1|a_1^*)p(\theta_2|a_2^*) - p(\theta_1|a_1^*)p(\theta_2|a_2))\right] = 0
\end{aligned}$$

where $c_2^*(\theta) \equiv \theta_1 + \theta_2 - c_1^*(\theta)$. Dividing both sides by $p(\theta_1|a_1^*)p(\theta_2|a_2^*)u_1'(c_1^*(\theta))$ and rearranging terms yield [22]

$$\frac{u_2'(c_2^*(\theta))}{u_1'(c_1^*(\theta))}\left\{1+\sum_{a_2}\mu_{a_2}\left[1-\frac{p(\theta_2|a_2)}{p(\theta_2|a_2^*)}\right]\right\} = \lambda + \sum_{a_1}\mu_{a_1}\left[1-\frac{p(\theta_1|a_1)}{p(\theta_1|a_1^*)}\right]. \tag{23}$$

By assumption A3 and the hypothesis of the Proposition, the right-hand side of condition (23) is nondecreasing in θ_1. If I can show that for all θ_2,

$$G(\theta_2) \equiv 1+\sum_{a_2}\mu_{a_2}\left[1-\frac{p(\theta_2|a_2)}{p(\theta_2|a_2^*)}\right] > 0$$

it will follow that $\frac{u_2'(c_2^*(\theta))}{u_1'(c_1^*(\theta))}$ is nondecreasing in θ_1, which in turn will imply that $c_1^*(\theta_1, \theta_2)$ is nondecreasing in θ_1 for each given θ_2.

First of all, $G(\theta_2) \neq 0$ for all θ_2, because then first-order condition (23) can not be satisfied. Secondly, either $G(\theta_2) > 0$ for all θ_2 or $G(\theta_2) < 0$ for all θ_2. If there are θ_2', θ_2'' with $G(\theta_2') < 0$ and $G(\theta_2'') > 0$, then it would imply that $u_2'(c_2^*(\theta))/u_1'(c_1^*(\theta))$ is nondecreasing in θ_1 for θ_2'' but is nonincreasing in θ_1 for θ_2', which is impossible by Lemma 8. Hence all I need is to prove $G(\theta_2) > 0$ for *some* θ_2.

Let $\bar{\theta}_2$ be the highest output level for agent 2. By assumption for each a_2 with $\mu_{a_2} > 0$, $g(a_2) < g(a_2^*)$. Then by assumption A3, $p(\bar{\theta}_2|a_2)/p(\bar{\theta}_2|a_2^*) \leq 1$. It follows that $G(\bar{\theta}_2) > 0$. This proves the desired result for agent 1. Similar argument will show that $c_2^*(\theta_1, \theta_2)$ is nondecreasing in θ_2. □

Proof of Proposition 5 Continued. To prove the last sentence of the main proof in the text, let λ be the multiplier for constraint (13) and $\mu(\theta_1)$ be the multiplier for (14) for each $\theta_1 \in \Theta_1$. The first order necessary condition of problem (15),[23] evaluated at $(\varepsilon = 0, \eta = 0, (\nu_k = 0))$ is given as:

$$\text{For } \varepsilon: \quad \delta - \sum_{\theta_1}\mu(\theta_1) = 0 \tag{24}$$

$$\text{For } \eta: \quad -\frac{u_1'(c_1^t)}{u_2'(c_2^t)} + \lambda = 0 \tag{25}$$

$$\text{For } \nu(\theta_1): \quad -\lambda\,\delta\,p_1(\theta_1) + \mu(\theta_1)\sum_{\theta_2}p_2(\theta_2)\frac{u_1'(c_1(\theta_1,\theta_2))}{u_2'(c_2(\theta_1,\theta_2))} = 0 \tag{26}$$

[22] This condition resembles the standard characterization of optimal wage contracts for the static principal-agent problem (see Holmstrom (1979) or Grossman and Hart (1983)). The difference here is that each agent's likelihood ratio enters into one side of the equation.

[23] The full rank constraint qualification condition holds for this problem. The Jacobi matrix of the constraint functions evaluated at the zero vector is essentially a diagonal matrix with positive diagonal elements.

Dividing equation (25) through by

$$\sum_{\theta_2} p_2(\theta_2) \frac{u_1'(c_1(\theta_1, \theta_2))}{u_2'(c_2(\theta_1, \theta_2))}$$

and summing over all θ_1 yields

$$\lambda\,\delta \sum_{\theta_1} \frac{p_1(\theta_1)}{\sum_{\theta_2} p_2(\theta_2) \frac{u_1'(c_1(\theta_1,\theta_2))}{u_2'(c_2(\theta_1,\theta_2))}} = \sum_{\theta_1} \mu(\theta_1). \tag{27}$$

Then using equations (24) and (25), one obtains the desired Eq. (12). □

Proof of Corollary 1. The result follows from equation (12) by Jensen's inequality if none of the limited commitment constraints is binding. Now consider a history at which agent 1's continuation utility $U_1(\sigma|h^t) = \underline{U}_1$. The variation method used in the proof of Proposition 5 should only allow a *reduction* in the *current* consumption of agent 1 and an *increase* in his *next-period* consumption, so that agent 1 will not be promised a continuation utility less than $\underline{U}_1$. This means that in Problem (15) the constraint $\varepsilon \geq 0$ (which is the variation in agent 1's lifetime utility $U_1(\sigma|h^t)$) should be added. Then the first-order condition (24) should hold with inequality:

$$\delta - \sum_{\theta_1} \mu(\theta_1) \leq 0.$$

Together with (25) and (27), this implies

$$\frac{u_2'(c_1^t)}{u_1'(c_2^t)} \leq \sum_{\theta_1} \frac{p_1(\theta_1)}{\sum_{\theta_2} p_2(\theta_2) \frac{u_1'(c_1^t(\theta_1,\theta_2))}{u_2'(c_2^t(\theta_1,\theta_2))}}$$

Jensen's inequality again implies the submartingale relation:

$$\frac{u_2'(c_1^t)}{u_1'(c_2^t)} \leq \sum_{\theta_1} \sum_{\theta_2} p_1(\theta_1) p_2(\theta_2) \frac{u_2'(c_2^t(\theta_1, \theta_2))}{u_1'(c_1^t(\theta_1, \theta_2))}.$$

However at this boundary $\underline{U}_1$ this type of argument can not establish that u_{1t}'/u_{2t}' will be a submartingale. □

Proof of Lemma 10. Suppose to the contrary such a history does not exist therefore by Proposition 6 agent 1 (weakly) prefers to save at all histories h^t, $t \geq 1$. It follows from the argument in Proposition 6 that for all h^t,

$$\frac{u_1'(c_1^t)}{u_2'(c_2^t)} \leq E_t \frac{u_1'(c_1^{t+1})}{u_2'(c_2^{t+1})}.$$

Given $\underline{c}_i = 0\ \forall i$, agents' consumptions are bounded from above by total output hence u_2' is bounded away from zero. Also, given that agent 1's expected utility is bounded within $(\underline{U}_1,\ Q)$ and $Q < \infty$, his consumption has to be bounded away from $\underline{c}_1 = 0$ hence u_1' is bounded from above. Therefore, the ratio u_{1t}'/u_{2t}' is a bounded submartingale hence converges almost surely. By the argument in Proposition 7, along almost all sample paths the contract converges to some first-best stationary contract, which is impossible. $\square$

Proof of Proposition 7. Part (a). I develop some necessary terminologies first. Fix an optimal contract $\sigma = (c_i, s_i)$. The pair of action plans (s_i) induces a probability space $(\Omega, \mathcal{F}, P)$, where $\Omega \equiv \Theta \times \Theta \cdots$ is the set of all infinite histories $\omega = (\theta^1, \theta^2, ...)$ and P is the product measure on Ω, induced by the action plans (s_i). Let X_t denote random variable u_{1t}'/u_{2t}'. Then $(X_t)_{t=1}^{\infty}$ is a submartingale with respect to the filtration $(\mathcal{F}_t)_{t=1}^{\infty}$ generated by $(X_t)_{t=1}^{\infty}$.

Note that (ii) means $P[X_t(\omega) \in [\alpha,\ \beta], \forall t] = 0$ for all $0 < \alpha < \beta < \infty$.

First, suppose there exist $0 < \alpha < \beta < \infty$ such that $P[X_t(\omega) \in [\alpha,\ \beta], \forall t] = 1$. Then (X_t) is a bounded submartingale therefore converges almost surely to some random variable Y. Since there are countable sample paths ω, the support of Y is countable. Consider some $y \in [\alpha, \beta]$ such that $P[A] = p > 0$ with $A = \{\omega \,|\, \lim_{t\to\infty} X_t(\omega) = y\}$, i.e. there is a positive measure of sample paths converging to y. For all $\epsilon > 0$, define event $A_t = \{\omega \,|\, X_\tau \in B_\epsilon(y), \forall \tau \geq t\}$ with $B_\epsilon(y) = (y - \epsilon, y + \epsilon)$. Then there is a $T = T(\epsilon)$ such that $P[A_T] > 0$. As

$$P[A_T] = P[X_T \in B_\epsilon(y)] \cdots P[A_t \mid X_\tau \in B_\epsilon(y), \forall T \leq \tau < t] > 0,$$

the conditional probability $P[A_t \mid X_\tau \in B_\epsilon(y), \forall T \leq \tau < t]$ must converge to 1. Therefore for almost all $\omega = (h^t, \theta^{t+1}, \cdots) \in A_T$, the sample paths of continuation process $(X_\tau, \tau > t \,|h^t)$ are almost surely bounded within $B_\epsilon(y)$ as $t \to \infty$. Since $A \subseteq A_{T(\epsilon)}$ for all $\epsilon > 0$, letting $\epsilon \to 0$ we have for almost all $\omega = (h^t, \theta^{t+1}, \cdots) \in A$, as $t \to \infty$, $(X_\tau, \tau \geq t \,|\, h^t)$ converges to y along almost all of the continuation sample paths from h^t. In other words, the continuation contracts $\sigma|h^t$ converge to the stationary first-best contract that entails constant marginal utility ratio equal to y.

Now suppose there exist $0 < \alpha < \beta < \infty$ such that $P[A] > 0$ with $A = \{\omega \,|\, \alpha \leq X_t(\omega) \leq \beta, \forall t\}$. Define the event $A_t = \{\omega \,|\, X_\tau(\omega) \in [\alpha,\ \beta], \forall \tau \geq t\}$. Clearly, $A \subseteq A_1 \subseteq \cdots$. Since for all t,

$$P[A] = P[\alpha \leq X_\tau \leq \beta] \cdots P[A_t \,|\, \alpha \leq X_\tau \leq \beta, \forall 1 \leq \tau < t] > 0,$$

it follows that the conditional probability $P[A_t \,|\, \alpha < X_\tau < \beta, \forall 1 \leq \tau < t]$ goes to 1 as $t \to \infty$. Therefore for almost all $\omega = (h^t, \theta^{t+1}, \cdots) \in A$, the continuation process $(X_\tau, \tau > t\,|h^t)$ is almost surely a bounded submartingale as $t \to \infty$. Then by the argument above, along almost all sample path $\omega \in A$ the continuation contracts converge to some stationary first-best contract.

Part (b). Since $\underline{c}_2 = 0$, Q is determined by Program Q that is defined by (4) and (5). Let $(\hat{a}_1, \hat{a}_2)$ be a solution to Program Q. Suppose there is an implementable action a_2' that is more productive than $\hat{a}_2$. Then by hypothesis $p(\cdot|a_2')$ first-order stochastically dominates $p(\cdot|\hat{a}_2)$. Therefore for each θ_1, $\sum_{\theta_2} u_1\left(\lambda(\theta_1+\theta_2)\right) p_2(\theta_2|a_2') > \sum_{\theta_2} u_1\left(\lambda(\theta_1+\theta_2)\right) p_2(\theta_2|\hat{a}_2)$. By Eqs. (4) and (5), agent 1's utility can be increased by replacing $\hat{a}_2$ with a_2', which contradicts the definition of Q.

Part (c). If $\underline{c}_j = -\infty$ then by definition (Section 2) $Q_i = (u_i(\infty) - g(\underline{a}_i))/(1-\delta)$, where $\underline{a}_i$ is agent i's minimum effort. Clearly if $Q_i < \infty$ and agent i's expected utility converges to Q_i then agent i's consumption c_i must diverge to ∞ and his effort must converge to the minimum level $\underline{a}_i$. $\square$

Proof of Proposition 8. I will use the following Lemma, which is a version of the Doob martingale convergence theorem.[24]

Lemma 12. *Let $(X_t)_{t=1}^{\infty}$ be a nonnegative submartingale. If $\lim_{t\to\infty} E(X_t) < \infty$, then there exists a random variable X such that $\lim_{t\to\infty} X_t = X$ almost surely and $E(X) \leq \lim_{t\to\infty} E(X_t)$. If further there exists some M such that $X_t < M$ almost surely for all t, then $E(X) = \lim_{t\to\infty} E(X_t)$.*

(a). Note that if only agent i has incentive problem, then the process $\{u_{jt}'/u_{it}'\}$ is a nonnegative martingale with $E(u_{jt}'/u_{it}') = E(u_{j1}'/u_{i1}')$ for all t. By the above Lemma it must converge almost surely to some random variable $X \geq 0$. Since incentive problem never disappears, by similar argument as in the proof of Proposition 7 it can be shown the probability that X belongs in some open interval $(\alpha, \beta) \subset \Re_+$ is equal to zero. It follows that $X = 0$ with probability one. Therefore agent i's consumption and expected utility diverge to $-\infty$ in the long run.

(b). For concreteness, suppose $M_1 < \infty$, $M_2 = \infty$. In this case marginal utility ratios $X_t \equiv u_{1t}'/u_{2t}'$ are uniformly bounded. By Lemma 12, X_t converges to some random variable X, and $E(X) = \lim_{t\to\infty} E(X_t)$. Since incentive problem never disappears, again by the same argument as in the proof of Proposition 7 it can be shown that for almost all sample path ω the limit $X(\omega)$ is either equal to M_1 or equal to 0. There are then two possibilities: either $X = M_1$ with probability one or it equals M_1 and zero with positive probabilities respectively. Which one of

[24] The first half of Lemma 12 is standard; see Billingsley [7] (p. 468). The second half follows from the fact that if the sequence (X_t) is uniformly bounded then it is uniformly integrable. See Fristedt and Gray [14] Theorem 20 (p. 480) for the convergence theorem and Definition 11 (p. 108) for uniform integrability.

these two possibilities will materialize depends upon the details of the model, in particular the configuration of incentive constraints. Suppose as in (b) that agent 2 is the only agent with moral hazard problem, then (X_t) is a martingale and the limit variable X satisfies $E(X) = \lim_{t\to\infty} E(X_t) = E(X_1) < M_1$, which implies that X takes on both M_1 and zero with positive probabilities.

(c). Now suppose M_1 and M_2 are both finite. Then both submartingales $(X_t = u'_{1t}/u'_{2t})$ and $(Y_t = u'_{2t}/u'_{1t})$ converge almost surely to some random variables X and Y respectively, with $E(X) = \lim_{t\to\infty} E(X_t) \geq E(X_1)$ and $E(Y) = \lim_{t\to\infty} E(Y_t) \geq E(Y_1)$. Again because incentive problem never dies out, the limit variable X will only take on two values: M_1 or $1/M_2$, and Y will only take on two values: M_2 or $1/M_1$. Moreover both X and Y must take on their two alternative values with positive probabilities respectively. The reason is simple: if $X = M_1$ with probability one then $Y = 1/M_1$ with probability one, which would contradict $E(Y) \geq E(Y_1) > 1/M_1$; similarly $Y = M_2$ is also impossible. So in this case both agents will have positive probabilities to become the winner who will consume all the output as $t \to \infty$. □

Proof of Proposition 9. By Corollary 1, u'_{1t}/u'_{2t} *is* a submartingale as long as agent 1's utility does not hit the boundary $\underline{U}_1$. Moreover the submartingale is bounded because agent 1's consumption must be bounded away from the lowest level $\underline{c}$ given that agent 1's continuation utility must lie in the finite interval $[\underline{U}_1,\ Q_1)$ (recall that Q_1 is the supremum of agent 1's lifetime utility).

Given the optimal contract, for each promised utility v for agent 1, let $\mathbf{R}(v)$ be the vector of marginal utility ratios u'_1/u'_2 in the next period and let $M(v)$ be equal to the maximum component of vector $\mathbf{R}(v)$, and let $W(v)$ be the corresponding continuation utility of agent 1. Starting from initial utility v_0 for agent 1, recursively define $v_t = W(v_{t-1})$, $r_t = M(v_{t-1})$, and $\mathbf{r}_t = \mathbf{R}(v_{t-1})$. Suppose agent 1's utility does not hit the boundary in finite time. Then we have a bounded monotone increasing sequence $(r_1, r_2, ...)$ with $r_t \leq E_t[\mathbf{r}_{t+1}]$ because u'_{1t}/u'_{2t} is a bounded non-degenerate submartingale. The sequence (r_t) must converge to some limit r^*, whence $(\mathbf{r}_t)$ converges to the vector whose components are all equal to r^*. The corresponding utility sequence $(v_0, v_1, ...)$ is bounded so it must have a subsequence that converges to some limit v^*. Together this would imply that for promised utility v^* to agent 1, the optimal contract entails a constant marginal utility ratio across current output signals. This contradicts the assumption that such first-best outcome is not possible.

If the boundary is not a reflecting barrier, then we would have an optimal contract that consists of a sequence of stationary one-period contracts, which is impossible by Corollary 2. □

References

[1] D. Abreu, D. Pearce, and E. Stacchetti, Toward a theory of discounted repeated games with imperfect monitoring, Econometrica 58 (1990), 1041-64.

[2] N. Al-Najjar, Incentive contracts in two-sided moral hazards with multiple agents, J. Econ. Theory 74 (1997), 174-95.

[3] A. Atkeson, International lending with moral hazard and risk of repudiation, Econometrica 59 (1991), 1069-89.

[4] A. Atkeson and R. Lucas Jr., On efficient distribution with private information, Rev. Econ. Stud. 59 (1992), 427-53.

[5] A. Atkeson and R. Lucas Jr., Efficiency and equality in a simple model of unemployment insurance, J. Econ. Theory 66 (1995), 64-88.

[6] S. Bhattacharyya and F. Lafontaine, Double-sided moral hazard and the nature of share contracts, Rand J. Econ. 26 (1995), 761-81.

[7] P. Billingsley, "Probability and Measure," John Wiley and Sons, New York, 1995.

[8] G. L. Clementi and H. Hopenhayn, A theory of financing constraints and firm dynamics, Quart. J. Econ. 121 (2006), 229-65.

[9] H. Cole and N. Kocherlakota, Efficient allocations with hidden income and hidden storage, Rev. Econ. Stud. 68 (2001), 523-42.

[10] J. Demski and D. Sappington, Optimal incentive contracts with multiple agents, J. Econ. Theory 33 (1984), 152-71.

[11] P. Diamond and J. Mirrlees, A model of social insurance with variable retirement, J. Pub. Econ. 10 (1978), 295-336.

[12] J. Farrell and E. Maskin, Renegotiation in repeated games, Games Econ. Behav. 1 (1989), 327-60.

[13] E. Friedman, Risk sharing and the dynamics of inequality, discussion paper 1235, Northwestern University, Center for Mathematical Studies of Economics and Management Sciences, November 1998.

[14] B. Fristedt and L. Gray, "A Modern Approach to Probability Theory," Birkhauser, Boston, MA, 1997.

[15] D. Fudenberg, B. Holmstrom and P. Milgrom, Short-term contracts and long-term agency relationships, J. Econ. Theory 51 (1990), 1-31.

[16] D. Fudenberg and J. Tirole, "Game Theory," MIT Press, Cambridge, MA, 1991.

[17] D. Fudenberg and J. Tirole, Moral hazard and renegotiation in agency contracts, Econometrica 58 (1990), 1279-1319.

[18] E. Green, Lending and the smoothing of uninsurable income, *in* "Contractual Arrangements for Intertemporal Trade," eds. E. Prescott and N. Wallace, pages 3-25. Minnesota Studies in Macroeconomics series, Vol. 1, University of Minnesota Press, Minneapolis, MN, 1987.

[19] S. Grossman and O. Hart, An analysis of the principal-agent problem, Econometrica 51 (1983), 7-45.

[20] S. Gupta and R. Romano, Monitoring the principal with multiple agents, RAND J. Econ. 29 (1998),427-42.

[21] B. Holmstrom, Moral hazard in teams, Bell J. Econ. 13 (1982), 324-40.

[22] S. K. Kim and S. Wang, Linear contracts and the double moral-hazard, J. Econ. Theory 82 (1998), 342-78.

[23] N. Kocherlakota, Implications of efficient risk sharing without commitment, Rev. Econ. Stud. 63 (1996), 595-609.

[24] E. Lazear and S. Rosen, Rank-order tournaments as optimum labor contracts, J. Polit. Econ. 89 (1981), 841-64.

[25] P. Legros and H. Matsushima, Efficiency in partnerships, J. Econ. Theory 55 (1991), 296 - 322.

[26] P. Legros and S. Matthews, Efficient and nearly-efficient partnerships, Rev. Econ. Stud. 60 (1993), 599-611.

[27] E. Ligon, Risk sharing and information in village economies, Rev. Econ. Stud. 65 (1998), 847-64.

[28] L. Ljungqvist and T. Sargent, "Recursive Macroeconomic Theory," The MIT Press, Cambridge, MA, 2000.

[29] C. A. Ma, Unique implementation of incentive contracts with many agents, Rev. Econ. Stud. 55 (1988), 555-72.

[30] J. Mirrlees, Notes on welfare economics, information and uncertainty, in "Essays in Equilibrium Behavior under Uncertainty," eds. M. Balch, D. McFadden, and S. Wu, North-Holland, 1974.

[31] D. Mookherjee, Optimal incentive schemes with many agents, Rev. Econ. Stud. 51 (1984), 433-46.

[32] C. Phelan and R. Townsend, Computing multi-period information constrained optima, Rev. Econ. Stud. 58 (1991), 853-881.

[33] C. Phelan, Incentives and aggregate shocks, Rev. Econ. Stud. 61 (1994), 681-700.

[34] C. Phelan, Repeated moral hazard and one-sided commitment, J. Econ. Theory 66 (1995), 488-506.

[35] C. Phelan, On the long run implications of repeated moral hazard, J. Econ. Theory 79 (1998), 174-91.

[36] D. Ray, The time structure of self-enforcing agreements, Econometrica 70 (2002), 547-82.

[37] W. Rogerson, Repeated moral hazard, Econometrica 53 (1985), 69-76.

[38] W. Rogerson, The first-order approach to principal-agent problems, Econometrica 53 (1985), 1357-67.

[39] S. Spear and S. Srivastava, Repeated moral hazard with discounting, Rev. Econ. Stud. 54 (1987), 599-617.

[40] N. Stokey, R. Lucas Jr., with Prescott, "Recursive Methods in Economic Dynamics," Havard University Press, Cambridge, MA, 1989.

[41] J. Thomas and T. Worrall, Income fluctuation and asymmetric information: an example of a repeated principal-agent problem, J. Econ. Theory 51 (1990), 367-390.

[42] J. Thomas and T. Worrall, Foreign direct investment and the risk of expropriation, Rev. Econ. Stud. 61 (1994), 81-108.

[43] J. Thomas and T. Worrall, Self-enforcing wage contracts, Rev. Econ. Stud. 55 (1988), 541-54.

[44] R. Townsend, Optimal multipleperiod contracts, and the gain from enduring relationships under private information, J. Polit. Econ. 90 (1982), 1166-86.

[45] R. Townsend, Risk and insurance in village India, Econometrica 62 (1994), 539-91.

[46] C. Wang, Dynamic insurance with private information and balanced budgets, Rev. Econ. Stud. 62 (1995), 577-96.

[47] C. Wang, Renegotiation-proof dynamic contracts with private information, Rev. Econ. Dynam. 3 (2000), 396-422.

[48] R. R. Zhao, Renegotiation-proof contract in repeated agency, J. Econ. Theory 131 (2006), 263-281.